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Brief Report

What We Say and How We Do: Action, Gesture, and Language in Proving

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In this Brief Report, we share the main findings from our line of research into embodied cognition and proof activities. First, attending to students' gestures during proving activities can reveal aspects of mathematical thinking not apparent in their speech, and analyzing gestures after proof production can contribute significantly to our understanding of students' proving practices, particularly when attending to dynamic gestures depicting relationships that are difficult to communicate verbally. Second, directing students to produce physical actions before asking them to construct a mathematical proof has the potential to influence their subsequent reasoning in useful ways, as long as the directed actions have a relationship with the proof content that is clearly meaningful to the students. We discuss implications for assessment practices and teacher education, and we suggest directions for future research into embodied mathematical proof practices.

Keywords: Embodied cognition; Gesture; Mathematical proof

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Mathematics classrooms contain a multitude of symbols, tools, and technologies that students and teachers engage with regularly. One particularly ubiquitous tool for communicating mathematical ideas that is growing in importance (e.g., Alibali, Boncoddo, & Hostetter, 2014; Alibali & Nathan, 2012) is gestures—the hand and arm movements that accompany speech. One type of mathematical communication in which gesture plays a key role is communication about justification and proof, which is "the basis of mathematical understanding and essential in developing, establishing, and communicating mathematical knowledge" (Stylianides, 2007, p. 289). Our two main claims are (a) that gestures are a powerful tool for mathematics education research because researchers can investigate students' mathematical thinking by attending to gesture and (b) that students' gestures can elicit new forms of mathematical reasoning. Although many scholars are already familiar with some elements of the first claim, the second claim rests on newly amassed evidence that gestures alter students' mathematical reasoning.

This line of research was inspired by observations of gestural communication in mathematics classrooms (e.g., Alibali & Nathan, 2012; Bieda & Nathan, 2009; Marrongelle, 2007; Rasmussen, Stephan, & Allen, 2004). Walkington et al. (2012) documented one example of a student struggling to distinguish between two very similar sounding words, *intersect* and *intercept* (Figure 1). The teacher supplemented her explanation with a gesture (Line 3), which prompted the student to realize that the teacher did not mean *intersect*. The student accompanied his description of *intercept* with a clearly different gesture, by crossing his two hands in an "X" (Line 6); through the coproduction of descriptive language and gesture, the two came to a shared understanding. This exchange was grounded in body-based spatial relations that were produced spontaneously and that allowed for the resolution of different mathematical definitions, illustrating one way that gesture is involved in communicating and meaning making about mathematical ideas.

Theories of grounded and embodied mathematical cognition (e.g., Barsalou, 2008; Lakoff & Nuñez, 2000) posit that bodily actions and multimodal language (e.g., gesture accompanying speech) independently serve as *grounding systems* (Nathan, 2014) for complex cognition. According to this perspective, our physical experiences with the world and our mental simulations of those experiences enact the internal processes traditionally labeled as "cognition." Actions and gestures accompanying speech can support mathematics learning (e.g., de Freitas & Sinclair, 2014; Edwards, Ferrara, & Moore-Russo, 2014), and gestures accompanying speech in particular are "involved not only in processing old ideas, but also in creating new ones" (Goldin-Meadow, Cook, & Mitchell, 2009, p. 271). For example, mentally simulated actions are important elements of learning mathematical ideas such as proportion (Abrahamson & Trninic, 2015; Martin & Schwartz, 2005) and algebra (Nathan, Kintsch, & Young, 1992), and gestures often reveal elements of mental simulations involved in mathematical reasoning that are not revealed in speech alone (e.g., Church & Goldin-Meadow, 1986).

1 T: So can you name a central angle on here? (* inaudible *).

So fill it in there. And then what's the intercept for that angle?

2 S: Is it BC?

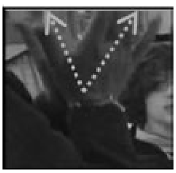
3 T: It is. **It's the arc that's inside that angle.**

Teacher makes a gesture of intercepted arc and moves the hands up and down: Please note that the gesture can only partially be seen

4 S: Okay see **I thought it meant like goin' through it** or hitting it.

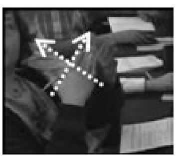
Student bends his elbow and moves his forearm forward

5 T: **No.**



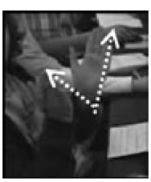
Teacher bends her elbow and repeats the gesture of intercepted arc

6 S: 'Cause like the lines **intercept**.



Student crosses his two hands in an "X"

That's where they meet. **It's not-**



Student, while looking at the teacher, reproduces the gesture of intercepted arc that the teacher made

7 T: Intersect. It's very close isn't it?

8 S: Sounds the same.

9 T: Yeah intercept means it's **inside the angle**.

Teacher repeats the gesture of intercepted arc and again moves the hands up and down

Figure 1. Gesturing to reach an understanding of the meaning of the mathematical term *intercept* (Walkington et al., 2012).

Embodied Transformational Proofs

Drawing upon grounded and embodied cognition perspectives, we examined gesture and speech as students produced proofs for conjectures. We classified individual student-produced proofs and considered such proofs as evidence for attributing specific proof schemes to the prover. According to Harel and Sowder (1998), the *transformational proof scheme* is one of the most sophisticated types of proof schemes that students possess because it "involve[s] operations on objects and anticipations of the operations' results" (p. 258). In particular, when students produce transformational proofs, they manipulate (i.e., transform) a mathematical concept or object by purposefully varying relationships, anticipating and observing the results, and subsequently deducing mathematical properties. As an example of a valid transformational proof, Williams et al. (2012) described a participant's reasoning when presented with the triangle inequality conjecture (Figure 2).

In Figure 3, the participant begins by establishing the base of the triangle (Line 1) as having a length of 5, which is arbitrary and used merely as a verbal starting point for his subsequent transformational proof. He then uses hand gestures to enact the movement of "bending" the two sides of the triangle towards each other to try to complete the triangle (Line 3), but the two sides (hands) fail to touch. These actions facilitate his reasoning that the conjecture must be true (Lines 4–5). In particular, the participant focuses on creating an argument that works for all triangles, using language and gesture together to communicate the generalizability of his justification.

This example illustrates our second claim: that gestures are a tool that can support students in their proof practices. Gestures can represent and transform mathematical objects such that students can observe and deduce general properties based upon their physical gestures and accompanying mental simulations. Below, we explore how analyses of gestures can act as a tool that enables researchers to capture information about mathematical reasoning beyond what can be gleaned from students' words alone and how directing participants to engage in certain actions can act as a tool that supports learners' subsequent proof practices.

Mary came up with the following conjecture: "For any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side." Provide a justification as to why Mary's conjecture is true or false.

Figure 2. The triangle inequality conjecture.

1 **Say one side's 5.**

Uses his fingers to indicate a general side length

2 **If the other two sides weren't at least 5 or even equal to 5,**

Uses his hands to indicate the other two sides, of unspecified lengths but shorter than the general side length indicated in Line 1

3 **If you tried to connect them, you think about it, they'd be too short to touch.**

Curves his hands in to illustrate the two shorter sides trying to close the triangle

4 **So they would have to—the sides would have to be longer**

Repeats the gesture of the two shorter sides failing to meet; re-invokes the original conjecture

5 **than the other remaining side.**

Repeats the gesture of the first general side length; completes answering the conjecture.

Figure 3. Gestures illustrate generalizability of justification.

Methodological Framing

We conducted a series of studies to understand the role that gesture plays in proving (e.g., Nathan et al., 2014; Walkington et al., 2014). Participants in the study highlighted here (n = 120) were undergraduates at a large Midwestern university who were given the triangle inequality conjecture and asked to prove or disprove the conjecture aloud without using pen and paper. To investigate how action influences thinking, we developed sets of actions designed to be mathematically relevant to the conjecture—that is, to support an embodied understanding of the properties of triangles—as well as parallel sets of actions designed to be mathematically irrelevant for the control condition (see Figure 4). Participants were directed to enact these relevant or irrelevant actions before addressing the conjecture. Some of the participants (n = 40) who performed mathematically relevant actions were informed prior to addressing the conjecture that the actions they were directed to enact were mathematically relevant to the conjecture in order to investigate the effect of prior awareness. Here, we focus specifically on the behaviors of participants who were not told in advance that the actions were related to solving the triangle conjecture (n = 40; Figure 2) or to our second conjecture about a system of gears that we will introduce momentarily (n = 40; Figure 6), so all further references to participants in this paper refer to these subsets.

Directed Action: Relevant

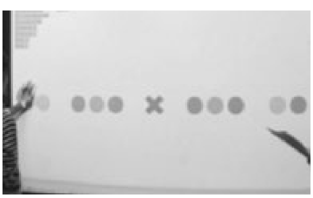
Graphic

Participants touched pairs of dots extending increasingly farther from the midpoint, thereby forming a series of triangles with their bodies until the final form, made with two parallel lines, was degenerate.

Description

40 participants performed only relevant actions and were told in advance that the actions related to the conjecture. All participants were asked to solve the problem a second time while explicitly thinking back to those actions.

Directed Action: Irrelevant



Graphic

Participants walked back and forth across the screen, tapping the dots one by one.

Description

(Prompts and hints not applicable to the irrelevant condition.)

Figure 4. Directed action conditions.

As the participant reads the triangle inequality conjecture aloud, she makes a nondynamic gesture as she says "triangle" for the first time:

In this gesture, the fingers of each hand represent a side, and the two thumbs together represent the third side.

Then she answers the conjecture with no gestures that evoke triangles:

1. **Mary's conjecture is true because if you look at the formula a squared plus b squared equals c squared, so you add two sides which would be a and b to equal c which is the hypotenuse. And… that's my final answer**.

When the participant is asked to explain her justification again, she makes another type of nondynamic gesture:

2. **Because—oh the triangle**

Two fingers meet to make a triangle vertex. Then each finger draws a side of the triangle.

- Mary's conjecture

Then both fingers swipe across each other to form the third side.

Note that although her hands are moving, the triangle she represents with her hands is nondynamic, and does not transform with her movements.

Then she continues, again answering the conjecture with no gestures that evoke triangles:

3. **that the addition of two lengths equals another length, um so if you look at the formula a squared plus b squared equals c squared, it—it's um for the triangle, measuring the sides.**

Figure 5. An invalid proof with two types of nondynamic gestures.

We coded participants' overall responses to the conjectures with respect to whether they had an appropriate intuition (i.e., demonstrated a belief in the truth of the conjecture; Zhang, Lei, & Li, 2016) and whether the proof was valid as indicated by a transformational proof scheme. We also coded properties of their speech and the gestures that they produced. In particular, we identified dynamic gestures, which can exhibit participants' enactment of the progressive transformation of mathematical entities. For example, in Lines 2 and 3 of Figure 3, the participant represents the two sides of the triangle in a dynamic gesture depicting the sides' changing relationship with the base as he tries to bring his fingers together to complete the triangle. In contrast, the participant in Figure 5 makes only nondynamic gestures during her invalid proof by forming and tracing a triangle but not manipulating any of the geometric properties of the triangle.

Our definition of dynamic gestures focuses not on the presence or absence of movement of the hands (cf. Göksun, Goldin-Meadow, Newcombe, & Shipley, 2013; Uttal et al., 2012) but on the dynamic nature of what the gesture depicts (e.g., Garcia & Infante, 2012; Marghetis, Edwards, & Nuñez, 2014). The dynamic gesture and accompanying language identifies the proof in Figure 3 as transformational, whereas the language and nondynamic gestures in Figure 5 exemplify an invalid proof.

Gestures and Language Analysis: Tools for Understanding Mathematical Reasoning

In our research, gesture coding began with two researchers who independently coded speakers' gestures from video clips using transcription analysis software (e.g., Transana, Version 2.52; Woods & Fassnacht, 2012), either with an a priori coding system or an emergent one. During data analysis sessions, we discussed the individual pairings of speech and gesture, reviewing the relevant video clip multiple times to arrive at a clearer picture of students' justification activities. We then created photo-enhanced transcripts to communicate contextual influences and referents of gestures and speech. We also performed statistical analyses investigating the associations between specific types of gestures (e.g., dynamic gestures) and relevant outcomes (e.g., proof validity).

In some cases, the information conveyed via gesture was redundant with the information in the speech (e.g., Figure 3, Line 3). In other cases, gestures and speech conveyed *distinct* information (e.g., Figure 3, Line 1) because people sometimes express mathematical ideas uniquely in gesture (Church & Goldin-Meadow, 1986; Pier et al., 2014). Dynamic gestures, in particular, often reveal information not expressed in speech because they involve the real-time transformation of mathematical entities that are difficult to articulate verbally.

Our research has established that certain speech characteristics are related to both proof accuracy and dynamic gesture production during proof. We ascertained this using two automated text analysis software programs, Coh-Metrix (Version 3.0; McNamara, Louwerse, Cai, & Graesser, 2013) and Linguistic Inquiry and Word Count (LIWC; Pennebaker, Booth, Boyd, & Francis, 2015). Coh-Metrix

An unknown number of gears are connected together in a chain. If you know what direction the first gear turns, how could you figure out what direction the last gear turns? Provide a justification as to *why* your answer is true.

(a)

Directed Action: Relevant

Graphic

Participants alternated between tapping a blue and yellow diamond placed an arm span apart with their palm.

(b)

Figure 6. The gear parity conjecture (a) and relevant directed action (b).

leverages natural language processing algorithms to measure characteristics of a text, such as its cohesion, lexical diversity, and syntactic complexity. LIWC is a dictionary-based program that calculates the percentage of words that fall into various dictionaries (e.g., words related to motion, time, and space would fall into a "relativity" dictionary). Using both resources in tandem enabled analysis of both the structure and content of participants' verbal justifications.

Following the researchers who developed these programs (e.g., Crossley, Allen, Kyle, & McNamara, 2014), we calculated correlation coefficients among the indices from each software program and a binary variable indicating whether a participant verbalized a transformational proof. The indices with significant correlations were entered into a stepwise logistic regression model predicting whether students produced transformational proofs. We also included demographic covariates (e.g., ACT or SAT math percentile) and whether they produced dynamic gestures. We found, unsurprisingly, that occurrences of *transformational speech* (i.e., "if…then" statements) predicted the likelihood of generating a valid proof (Pier et al., 2016). It is important to note that we also found that dynamic gesture production independently predicted the likelihood of producing a transformational proof (Pier et al., 2014), suggesting that elements of mathematical reasoning were conveyed in gesture that were not conveyed in speech.

In conclusion, researchers can leverage fine-grained analyses of speech and gestures to construct grounded hypotheses about how students' reasoning evolves as they grapple with different mathematical ideas. Examining the redundant, unique, and complementary information provided by gestures can allow for a more complete and generative picture of students' reasoning, justification, and proof practices.

Directed Action: Tools for Influencing Mathematical Reasoning

Thus far, we have considered how mathematics education researchers can unpack gestures along with speech to serve as tools for understanding proof practices. In this section, we show how directed actions can serve as a tool to influence learners' mathematical intuition and proof production.

The logistic regression models predicting intuition and validity showed that performing relevant directed actions increased the likelihood that participants would arrive at the intuition necessary to solve the problem (e.g., a sense that the conjecture was true based on properties of triangles). However, performing relevant directed actions did not increase the likelihood that participants would verbalize valid transformational proofs (Nathan et al., 2014). After the participants made their first proof attempts, we explicitly told them that the actions they had performed were related to the conjecture and asked them to again attempt a proof for the triangle inequality conjecture. Twenty-five percent of the participants who had initially produced an invalid proof subsequently produced a valid transformational proof. This result was consistent with prior research showing that explicitly stating such relationships helps with transfer (e.g., Anderson, Corbett, Koedinger, & Pelletier, 1995).

However, contrary to this prior research, directed actions designed to accompany a nongeometric conjecture—one about parity in a system of gears (Figure 6a)—produced strikingly different results. When participants were explicitly told that the directed action (Figure 6b) was related to the conjecture, nearly 18% of the participants shifted from a valid transformational proof about the gears to an invalid proof. This result seemed to be driven by the relatively difficult-to-grasp relationship between the relevant actions (alternation) and the key idea behind the proof (parity).

The conflicting results between the triangle and gear conjectures suggest that directing learners to perform actions may facilitate proof production only if the actions can be meaningfully connected to the relevant mathematical relationships. The directed actions for the triangle inequality conjecture required participants to form triangles with their bodies, which readily connected their actions to the conjecture. The directed actions for the gear conjecture, however, required participants to enact the more abstract concept of alternation, and the relevance of this abstract action was more difficult to discern. When participants were informed about the connection, they often attempted to connect their actions to the conjecture in an incorrect or inappropriate way.

Conclusions

Our research to date has yielded two key conclusions regarding the influential roles of action and gesture in mathematical thinking. First, attending to students' gestures in tandem with language can contribute significantly to understanding students' mathematical reasoning and proving practices, particularly when attending to dynamic gestures depicting relationships that may be difficult to communicate verbally. Second, directing students to produce physical actions before asking them to construct a mathematical proof has the potential to influence their subsequent reasoning in both productive and unproductive ways, showing an important connection between mathematical reasoning and bodily movements.

Directed actions facilitate proofs when such actions can be readily connected to the relevant mathematical relationships.

Given these findings, we suggest directions for future research, assessment practices, and teacher education. First, the results reported here and elsewhere (e.g., Abrahamson & Trninic, 2015; Ottmar, Landy, & Goldstone, 2012; Smith & Neumann, 2014) show that there is a need to understand more fully how directed actions carried out by learners influence their mathematical reasoning and, more generally, how actions shape cognition. Second, analyses showing the independent contributions of gesture and speech in rich accounts of students' mathematical reasoning suggest the need for assessments that capture embodied ways of knowing as a complement to current assessment methods that are predominantly verbal and written in nature. Third, because gestures reveal insights about student thinking, we need to better understand how teachers and researchers can and do make use of this information in real time in order to shape subsequent instruction. Attending to student gestures as diagnostic of their mathematical understanding may well have a place in teacher education and in-service professional development programs.

This line of research developed as an outgrowth of our ethnographic classroom work, highlighting the challenges of translating observations of spontaneous action and gesture into effective pedagogical interventions. Before integrating interventions into mathematics classrooms and mathematics teacher education, we need to further refine our understanding of precisely when and how considerations of the body support and reveal students' mathematical thinking. Consequently, we encourage researchers to consider designing and eliciting specific actions that manifest mathematical ideas in order to support students' reasoning and to examine both spoken language and gesture when analyzing records of student reasoning. In particular, directing learners' actions and attending to their spontaneous gestures are powerful tools for both supporting and illuminating proof practices.

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