

Making Connections in Math: Activating a Prior Knowledge Analogue Matters for Learning

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This study investigated analogical transfer of conceptual structure from a prior-knowledge domain to support learning in a new domain of mathematics: division by fractions. Before a procedural lesson on division by fractions, fifth and sixth graders practiced with a surface analogue (other operations on fractions) or a structural analogue (whole number division). During the lesson, half of the children were also asked to link the prior-knowledge analogue they had practiced to fraction division. As expected, participants learned the taught procedure for fraction division equally well, regardless of condition. However, among those who were not asked to link during the lesson, participants who practiced with the structurally similar analogue gained more conceptual knowledge of fraction division than did those who practiced with the surface-similar analogue. There was no difference in conceptual learning between the two groups of participants who were asked to link; both groups performed less well than did participants who practiced with the structural analogue and were not asked to link. These findings suggest that learning is supported by activating a conceptually relevant prior-knowledge analogue. However, unguided linking to previously learned problems may result in negative transfer and misconceptions about the structure of the target domain. This experiment has practical implications for mathematics instruction and curricular sequencing.

People learn new information in the context of their own prior knowledge. For example, when learning new mathematical concepts, students draw on their existing knowledge of related mathematical concepts and procedures. Understanding how learners build on prior knowledge is crucial to understanding how cognitive development occurs. Moreover, understanding how best to build on what learners already know is at the heart of effective instruction.

Understanding how new knowledge builds on prior knowledge is closely related to issues of analogical transfer. Transfer occurs when a learner uses previously learned knowledge to support new learning or problem solving. There are two aspects to successful transfer: *identification* of an appropriate analogue and *adaptation* of relevant aspects of that analogue (see Gick & Holyoak, 1987). When both of these processes are accomplished successfully, positive transfer occurs, benefiting subsequent performance or learning. This analysis suggests that transfer can go wrong in one of two ways. First, a learner could fail to identify a helpful analogue, resulting in no transfer, or the learner could identify a misleading analogue for transfer, resulting in transfer of knowledge that is ill-suited for the target problem, termed negative transfer. Second, a learner

might identify a helpful analogue yet be unable to transfer the relevant information appropriately. Many factors affect learners' ability to transfer, including features of the learner (e.g., Nokes & Belenky, 2011; Novick, 1988), the problems (e.g., Bassok, 1996; Chen & Daehler, 1989; Daehler & Chen, 1991; Novick, 1988), and the context (e.g., Gick & Holyoak, 1980; Novick & Holyoak, 1991; Reed, Dempster, & Ettinger, 1985; Richland & McDonough, 2010; Thompson & Opfer, 2010).

Obstacles to Analogical Transfer

Gentner's (1983) structure-mapping theory outlines the characteristics of a strong analogue. Within this framework, *surface* (or perceptual) features are attributes that describe items in a system, and *structural* features are attributes of important relationships within a system (e.g., causal or spatial relationships). When learners map across surface features, prior knowledge about the first system is not necessarily helpful for understanding the second system. In contrast, when learners map across structural features, prior knowledge about the first system is relevant for transfer.

Even when learners have prior knowledge of relevant structural analogues, they do not always spontaneously map from these analogues to novel problems. Learners face several obstacles when attempting to identify a helpful source analogue and to map and adapt knowledge from source to target problems (e.g., Catrambone & Holyoak, 1989; Gick & Holyoak, 1980; Novick, 1988; Novick & Holyoak, 1991; Reed et al., 1985).

Analogue identification. To identify a supportive analogue that shares structural features with a target problem, a learner must perceive the structural features of each problem. This may be straightforward for experts in a given domain, who readily perceive structural similarities, but it is often challenging for novices, who are more likely to focus on surface attributes (Chi, Feltovich, & Glaser, 1981). For example, Novick (1988) found that math novices were relatively poor at identifying structural analogues for math problems, particularly in the face of a perceptually distracting alternative. Moreover, when no structural analogue was available, experts as well as novices often transferred from a surface analogue. Surface analogues can have a powerful effect, leading in many cases to negative transfer of incorrect procedures.

Young children seem to be vulnerable not only to perceptually distracting analogues, but also to recently learned analogues. Chen and Daehler (1989) succeeded in training 6-year-old children to generate a solution procedure by generalizing across structurally similar problems. However, once children learned a procedure, they tended to apply it to target problems with equal frequency, regardless of whether the target problem shared features with the source analogue. Thus, interference in choosing the right analogue seems to come from both perceptually distracting as well as recently learned analogues.

Mapping and adaptation. Even when structural analogues are available, learners face the challenge of recognizing that an analogue is actually applicable as well as the challenge of adapting solution procedures appropriately. Learners often need support, such as generic hints that an analogue is useful (Gick & Holyoak, 1980) or opportunities to refer back to analogue problems (Reed et al., 1985), in order to transfer their knowledge.

Even when provided with such support, some learners still struggle with transfer. Novick and Holyoak (1991) gave participants explicit information about analogous aspects of a source and

target problem to facilitate transfer. Participants who received this information were more likely to transfer than were those who received only a retrieval hint or no hint at all. However, transfer rates were still low—around 50% in the mapping-aid condition—presumably due to the difficulty of adapting the procedure, despite proper mapping. This difficulty may derive from learners lacking a good conceptual understanding of why the original procedure resulted in the correct answer. Supporting this view, Gentner, Loewenstein, and Thompson (2003) greatly improved participants' positive transfer to a target problem by allowing them to first compare two structural source analogues and identify aspects of shared conceptual structure. Understanding the conceptual structure of an analogue may be important for successfully transferring solution procedures to target problems.

This work on analogical mapping has two important implications. First, learners may need explicit instructions to retrieve an analogue and search for shared structure. Second, learners may be better able to adapt their knowledge of a source domain when they understand the conceptual structure of that domain. We next consider the importance of conceptual knowledge in mathematics and its potential role in supporting new learning through analogical transfer.

Transfer of Conceptual and Procedural Knowledge

Many accounts of mathematical knowledge distinguish between conceptual and procedural knowledge (e.g., Baroody & Dowker, 2003; Hiebert & Lefevre, 1986; Rittle-Johnson & Siegler, 1998). Procedural knowledge is commonly defined as knowledge of sets of actions that can be used to solve a particular type of problem. In contrast, conceptual knowledge is defined as knowledge of principles that apply within a domain, and knowledge about relationships among elements within a domain, including knowledge of the meanings of structures and processes used in the domain (Rittle-Johnson, Siegler, & Alibali, 2001).

Conceptual knowledge supports new learning in domains that have not yet been mastered (e.g., Byrnes & Wasik, 1991; Hecht & Vagi, 2010, 2011) and helps learners generate procedures for solving novel problems (e.g., Rittle-Johnson & Alibali, 1999). Because conceptual knowledge supports procedural knowledge, learners who have a conceptual understanding of a base analogue and its associated procedures may be better at adapting relevant procedures to a new problem compared with learners whose knowledge is limited to procedures only. Moreover, comprehending the conceptual basis of procedures may also aid learners in identifying which types of problems are strong analogues for procedural transfer.

Furthermore, with only a few recent exceptions (Day & Goldstone, 2011; Thompson & Opfer, 2010), the vast majority of studies of analogical transfer have focused on procedural knowledge and have failed to consider whether conceptual knowledge is spontaneously transferred across analogous problems. In most studies of analogical transfer, participants are taught a procedure for a source problem and then are tested on whether they apply that procedure to a target problem. Because the tasks are often novel and the conceptual underpinnings of the problem domains are rarely assessed or taught, there has been little opportunity to consider transfer of conceptual knowledge.

We propose that children may be able to transfer their conceptual understanding of a previously mastered mathematical domain to a novel, structurally similar domain. Thus, we suggest that analogical transfer from a conceptually rich prior-knowledge domain may be one way of

supporting the acquisition of conceptual knowledge in mathematical domains that are traditionally challenging for students.

Target Domain: Division by Fractions

In this study, we investigate analogical transfer as a method of supporting the acquisition of conceptual knowledge in one such challenging domain: fraction division. Division by fractions is widely recognized as an important (e.g., National Governors Association Center for Best Practices [NGA Center] & Council of Chief State School Officers [CCSSO], 2010; National Mathematics Advisory Panel, 2008) yet difficult topic to master (e.g., Ma, 1999; Siegler, Thompson, & Schneider, 2011). Children's errors on fraction division problems suggest negative transfer from other operations on fractions, specifically, addition and subtraction (i.e., operating on only the numerators when the fraction denominators are equal; Siegler et al., 2011). These errors strongly suggest that children spontaneously, but inappropriately, transfer solution procedures across fraction problems with different operational structures. Children may identify other types of fraction problems as helpful analogues for transfer in part because of perceptual similarities and in part because of recency. When students encounter fraction division, they are likely to have recently encountered fraction addition and subtraction, as fraction operations are grouped together in many textbooks (e.g., Billstein, McDougal Littell Inc., & Williamson, 1999; Burton & Maletsky, 1998; Charles, Dossey, Leinwand, Seeley, & Vonder Embse, 1999; Clements, 1998; University of Chicago School Mathematics Project, 2007). In this study, we consider other fraction operations to be surface-similar analogues for fraction division. Transferring procedural knowledge from other fraction operations may yield negative transfer, and may thus hinder learning.

In addition to struggling with fraction procedures, American students and teachers often lack conceptual understanding of what it *means* to divide by a fraction (e.g., Ma, 1999). There are many aspects of conceptual knowledge of fraction division, including knowledge of what a fraction is, an understanding of the operational structure of division, and conceptual models of the relationships between quantities when dividing.

In this study, we focus on children's conceptual models of the relationships between quantities in fraction division, as measured by their abilities to generate and comprehend story contexts and visual representations for fraction division. This focus is in line with the Common Core State Standards for Mathematics (NGA Center & CCSSO, 2010), with the Institute of Education Science (IES) practice guide for fraction instruction (Siegler et al., 2010), and with prior research on conceptual understanding of fraction division (e.g., Ma, 1999). Whole number division provides a structurally similar conceptual analogue to fraction division, because problems in these domains share a division relationship between quantities. In particular, children's informal mental models of division as sharing, grouping, and partitioning (e.g., Correa, Nunes, & Bryant, 1998; Steffe & Olive, 2010) as well as the language used to create whole-number division story problems can be mapped to and adapted for fraction division.

Our choice of whole number division as a structural analogue is closely tied to our goals for conceptual learning in this study, specifically, transfer of knowledge of the operational structure of division by whole numbers to division by fractions. Of course, children's conceptual knowledge has many facets, and some facets of conceptual knowledge of whole number division do not apply to fraction division, such as the notion that "division makes smaller" (e.g., Fischbein,

Deri, Nello, & Mariono, 1985). Furthermore, conceptual knowledge of fraction magnitudes is critical for all types of fraction problems but does not support conceptual understanding of particular fraction operations. In this study, our identification of other operations on fractions as surface analogue domains and whole number division as a structural analogue domain is critically tied to our practical goals for this study: inhibiting negative transfer from other fraction procedures to fraction division equations and supporting positive transfer from conceptual models of whole number division to fraction division concepts. Our findings may provide evidence that appropriate analogous problems can support transfer of conceptual knowledge.

A second goal of this study was to examine analogical transfer using an ecologically valid prior-knowledge domain. Most experimental studies of analogical transfer examine learning with logic problems that are not typically part of participants' prior experience. Few studies have investigated transfer of knowledge learned outside of an experimental setting (but see Dunbar, 2001) or from children's own prior knowledge (but see Thompson & Opfer, 2010). When learning fraction division, children seem to transfer knowledge from other domains, but not always in intended ways. An analogical transfer framework may be useful for understanding aspects of children's performance that might reflect positive, negative, or unsuccessful transfer during learning in this important and challenging mathematical domain.

The Current Study

In brief, this study had two major goals. First, we examine whether key findings from the analogical transfer literature hold when the source domain is an ecologically valid domain, based in children's prior knowledge, built through formal instruction. Specifically, we test whether children learn more about fraction division when drawing on a structural analogue than when drawing on a surface-similar analogue, and whether they need explicit links to adapt their rich prior knowledge.

Second, we investigate children's abilities to transfer *conceptual knowledge* of a prior-knowledge analogue domain to a target domain. Most past research on analogical transfer has focused on the conditions under which learners retrieve and adapt procedures, without considering learners' conceptual understanding.

We hypothesized that children who receive practice problems with whole number division prior to learning about fraction division will gain more conceptual knowledge about fraction division than will those who practice with other types of fraction problems. Further, we hypothesized that children who are explicitly asked to make links between the source and target domains will show greater transfer of conceptual knowledge from the analogue domain to division by fractions than will children who are not explicitly directed to make links. Given that there are bidirectional relationships between conceptual and procedural knowledge (e.g., Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001), we expect to see gains in conceptual learning in simple fraction division scenarios as well as differences in children's abilities to adapt a learned procedure to complex problems.

METHOD

Participants

The sample included 100 children (68 boys and 32 girls). Participants were recruited in the summer after fifth grade ($n_{5+} = 39$), during sixth grade ($n_6 = 29$), and in the summer after sixth

grade ($n_{6+} = 32$).¹ Most were recruited through elementary and middle schools in an urban district in the Midwestern United States, using letters sent home in students' backpacks from school. A small subset of participants was recruited from a list of families in the same community who had indicated interest in participating in studies in our lab. Parents or guardians reported their child's race and ethnicity on a demographic questionnaire: 69% were identified as White, 11% as Asian, 5% as Black or African American, 5% as Hispanic or Latino, 5% as being of Mixed racial background, and 1% as Native American; 4% chose not to respond.

Eleven children were at or near ceiling at pretest, scoring 100% on the procedural scale and 100% on the conceptual story items, the conceptual picture items, or both. These children were excluded from the analyses as they had no or little room to improve from the lesson. Thus, the final sample includes 89 participants, with 58 boys and 31 girls in post-fifth grade ($n_{5+} = 35$), sixth grade ($n_6 = 27$), and post-sixth grade ($n_{6+} = 27$).

Design and Procedure

All participants completed a pretest, one of two analogue worksheets, an oral lesson either with or without links, and a posttest during a single session. The analogue worksheets contained either fraction addition and subtraction problems or whole number division problems, and the oral lesson either included explicit links to the analogue worksheet or did not include such links. These two factors, analogue type and links, were crossed in a 2×2 between-subjects design, resulting in four conditions. Children were randomly assigned to one of the four groups: division analogue with links ($n = 24$), division analogue with no links ($n = 21$), fraction analogue with links ($n = 21$), or fraction analogue with no links ($n = 23$).

All sessions took place in a lab setting with a female experimenter. Participants were first asked to complete the pretest and were told they would receive a lesson about division by fractions. Immediately prior to the lesson, the experimenter gave participants either the whole-number division or operations on fractions analogue worksheet, referring to the problems as "warm-up problems." In the no-links conditions, the experimenter removed the analogue worksheet and proceeded with a scripted oral lesson designed to teach children a procedure for dividing by fractions. In the links conditions, children were allowed to keep the analogue worksheet and were asked periodically to refer back to it. Upon completion of the lesson, participants were given the posttest. The lesson included only examples with unit fractions (i.e., fractions with numerators of 1). Participants took 45 min to 1 hr to complete the entire study.

Materials

Procedural scale. The pretest and posttest included a procedural scale of six items designed to assess children's procedural knowledge of division by a unit fraction. See Table 1 for examples, rationales, and descriptions of the items. The procedural scale included four equation items requiring children to solve fraction division problems. Equation items were scored as correct if a child demonstrated a procedure that would result in the correct answer, even if the

¹In the local school district, students begin learning about symbolic operations on fractions in fifth grade and are expected to learn fraction division between sixth grade and eighth grade (Madison Metropolitan School District, 2009).

TABLE 1
Description of Pretest and Posttest Items

<i>Item type</i>	<i>Example</i>	<i>Rationale</i>	<i>Pretest</i>	<i>Posttest</i>	<i>Scoring</i>
Solve a fraction division equation with a fraction division equation	$2 \div \frac{1}{3} =$	<i>Procedural Knowledge Items</i> Evaluate knowledge of procedure for fraction division	4	4	1 point each, for using a correct procedure
Generate a story to correspond with a fraction division equation	Write a story to represent $5 \div \frac{1}{7}$	<i>Conceptual Knowledge Items</i> Assess whether S understands meaning of division by fractions and can apply in real-world setting	2	2	Setup: 1 point each for a story that represents the given phrase Question: 1 point each for a story that represents the correct answer
Select a model to correspond with a fraction division equation	Which model best represents $4 \div \frac{1}{2}$	Assess whether S understands meaning of division by fractions and can identify appropriate visual representation of quantities and relational structure	2	2	1 point each, for correct choice of model
Assess and justify a hypothetical student's procedure	Drew is a student at another school... is this strategy correct?	<i>Procedure Justification Items</i> Assess whether S can identify correct procedures	2	2	1 point each for procedure accurately identified as correct or incorrect
Solve fraction division equation with a complex divisor and/or dividend (<i>procedural transfer</i>)	$6\frac{2}{5} \div 4\frac{1}{8} =$	<i>Transfer Items</i> Evaluate generalization of procedure for fraction division to problems more complex than those presented in the lesson	0	2	1 point each, for using a correct procedure
Generate a story to correspond with fraction division equation in which divisor is larger than dividend (<i>conceptual transfer</i>)	Write a story to represent $\frac{3}{5} \div \frac{4}{7}$	Assess whether S understands and can generalize the meaning of division by fractions and can apply in real-world setting	0	2	Setup: 1 point each for a story that represents the given phrase Question: 1 point each for a story that represents the correct answer
Select a model to correspond with a fraction division equation in which the divisor is larger than the dividend (<i>conceptual transfer</i>)	Which model best represents $\frac{1}{3} \div \frac{2}{5}$	Assess whether S understands and can generalize meaning of division by fractions and can identify appropriate visual representation of quantities and relational structure	0	2	1 point each, for correct choice of model

child made a simple calculation error. Participants received 1 point for each correct item. The procedural scale also included two procedure evaluation items, for which participants evaluated hypothetical children's procedures as correct or incorrect. Procedure evaluation items were scored as correct if the participant's evaluation was valid, and participants received 1 point for each correct answer.

Including all six items, the procedural scale had good reliability as measured by Cronbach's alpha (Cronbach, 1951) at pretest ($\alpha = .87$) and posttest ($\alpha = .84$). Furthermore, the removal of any particular item did not greatly affect reliability. Therefore, the procedure evaluation and equation items were combined to create total pretest and posttest procedural scores using a weighted average, such that equation items and evaluation items were weighted equally.

Conceptual scale. The pretest and posttest also included a conceptual scale of five items designed to assess children's conceptual understanding of division by a unit fraction. See Table 1 for examples, rationales, and descriptions of these items as well. The conceptual scale included two story items, similar to those used in Ma (1999) and recommended by the Common Core State Standards (NGA Center & CCSSO, 2010), in which participants were asked to write a story to represent a given expression. The conceptual story items received maximum points if the participant appropriately represented the given equation. To illustrate, for the expression $6 \div \frac{1}{2} =$, a correct story might represent 6 as a given quantity (e.g., of cookies), represent $1/2$ as the value of a segment (e.g., one serving is half of a cookie), and ask how many segments there are in the given quantity (e.g., How many servings of cookie?). Each story response received 1 point for a correct setup and 1 point for a correct final question. To assess reliability, two independent coders coded the accuracy of 100 stories (1 from each participant); the coders agreed on 100% of trials. The conceptual scale also included three picture model items requiring students to match a fraction division expression to a visual representation (see Table 1). These items were based on whole-number division practice problems found in some traditional elementary math textbooks, and they are aligned with recommendations made by the Common Core State Standards for Mathematics for visual representations of fraction division (NGA Center & CCSSO, 2010). The picture items were multiple-choice and were scored as correct if the correct representation was chosen. Participants received 1 point for each correct choice.

Including all 7 points from the five items, the conceptual scale also had good reliability as measured by Cronbach's alpha (Cronbach, 1951) at pretest ($\alpha = .81$) and posttest ($\alpha = .77$). Furthermore, the removal of any particular item did not greatly affect reliability. Thus, scores on the story and picture items were combined to create total pretest and posttest conceptual scores, also using a weighted average, so that both story and picture items were weighted equally.

Transfer items. Finally, the posttest also included several procedural and conceptual far-transfer items. Unlike the lesson, pretest, and posttest problems, which contained only whole numbers divided by unit fractions, the transfer items contained more complex operands (i.e., mixed numbers and other proper fractions). Thus, these problems assessed children's abilities to flexibly adapt what they had learned about unit fractions. The transfer test included two procedural transfer items, and they were scored in the same manner as the other equation items. Two story items and two picture items were included to assess conceptual transfer, and they were scored in the same manner as the other story and picture items. In contrast to the items included in the conceptual scale, the transfer story items were uncorrelated with the transfer picture items ($-.10 < r_s < .13$), and therefore, the transfer story items and picture items were

analyzed separately, rather than combined into a conceptual transfer score. Examples of these items can be found in Table 1.

Analogue worksheet. During the sessions, the analogue worksheets were referred to as “warm-up problems” to cue their relevance to the lesson. These worksheets were designed to simulate practice problems that appear at the beginning of textbook lessons, when children are often asked to practice recently learned material. Each analogue worksheet contained three equation procedural items and one conceptual item (see Appendix A). The procedural items on the division analogue worksheet were whole number division problems (e.g., $36 \div 6 =$); those on the fraction analogue worksheet were fraction addition and subtraction problems (e.g., $\frac{1}{3} + \frac{1}{6} =$). The conceptual item on the division analogue worksheet asked participants to visually represent a whole number division equation; this item was intended to target conceptual understanding of division. The conceptual item on the fraction analogue worksheet asked participants to visually represent a relationship between two fractional quantities, $\frac{1}{2}$ and $\frac{1}{4}$; this item was intended to target conceptual understanding of fraction magnitude because it supports students’ reasoning in all fraction operation domains. A final question on each worksheet asked participants to consider similarities between the analogue and a target problem, even though participants had not yet learned about the target problem, to signal the relevance of these problems to the lesson.

Oral lesson. The complete script of the oral lesson and the lesson worksheet are presented in Appendix B. The lesson was couched in a division-by-fractions word problem. During the lesson, the experimenter showed participants how to represent the word problem symbolically, demonstrated invert-and-multiply, and supervised participants as they practiced three target problems. Participants were given a worksheet on which to practice during the lesson as well as scripted, correct answers following all responses. To demonstrate invert-and-multiply, the experimenter showed participants how to invert a fraction and provided them with examples on the lesson worksheet. Then, the experimenter gave a short explanation of multiplying two fractions, in case the child had not yet learned this procedure, and allowed the child to practice on the lesson worksheet with feedback. Finally, the experimenter demonstrated how invert-and-multiply could be applied to solve the original word problem, referring to steps displayed on the lesson worksheet. Participants were asked to give a reason for each step in the procedure but did not receive feedback on their reasons. At the end of the lesson, participants were given practice problems to solve using invert-and-multiply on unit fractions and received feedback on their answers. In the links conditions, participants were also asked to refer back to their analogue worksheet and compare those “warm-up problems” to division by fractions at five points during the lesson. No feedback was provided about participants’ responses to the linking prompts. See Appendix B for the script of the entire lesson, including the linking prompts (used only in the links conditions), which are underlined.

During the lesson, children occasionally asked questions about the mathematical content or about the task. In such cases, the experimenter repeated information from the script to address children’s questions. Some participants also expressed their uncertainty about their responses. In these cases, the experimenter told the child that it was “OK” to be unsure, and that they would continue the lesson. All sessions were videotaped to ensure consistency and were reviewed by a coder who was blind to analogue condition. Overall, there were few deviations from the script, and they occurred equally infrequently across conditions.

RESULTS

On the procedural scale, participants scored an average of 41% correct at pretest ($M = 0.41$, $SD = 0.33$) and 73% correct at posttest ($M = 0.73$, $SD = 0.29$). On the conceptual scale, participants scored an average of 18% correct at pretest ($M = 0.18$, $SD = 0.22$) and 33% correct at posttest ($M = 0.33$, $SD = 0.29$). Pre-post difference scores on the procedural and conceptual scales were examined in a series of 2 (analogue: whole number division or operations on fractions) \times 2 (links: links or no links) between-subjects analyses of covariance, with grade level (treated continuously) and gender as covariates.

Procedural Learning

Participants in all four conditions received a lesson about how to divide by a fraction as well as guided practice with the procedure. Therefore, we anticipated that participants in all conditions would learn the procedure, and they did so equally well. There were no effects of analogue, $F(1, 83) = 0.07$, *ns*, $\eta_p^2 < .01$, linking, $F(1, 83) = 1.79$, *ns*, $\eta_p^2 = .02$, or their interaction, $F(1, 83) = 0.46$, *ns*, $\eta_p^2 = .01$, on procedural gains ($M_{\text{DivisionLink}}[M_{\text{DL}}] = 35\%$, $M_{\text{DivisionNoLink}}[M_{\text{DnL}}] = 30\%$, $M_{\text{FractionLink}}[M_{\text{FL}}] = 37\%$, and $M_{\text{FractionNoLink}}[M_{\text{FnL}}] = 24\%$ improvement). Furthermore, there were no significant effects of grade, $F(1, 83) = 0.51$, *ns*, $\eta_p^2 = .01$, or gender, $F(1, 83) = 0.76$, *ns*, $\eta_p^2 = .01$.

Conceptual Learning

On the conceptual items, we expected participants in the division analogue condition to improve more than participants in the fraction analogue condition, and we expected those asked to link to improve more than those not asked to link. Unexpectedly, the main effect of analogue was nonsignificant, $F(1, 83) = 0.26$, *ns*, $\eta_p^2 < .01$, as was the main effect of linking, $F(1, 83) = 0.66$, *ns*, $\eta_p^2 = .01$. However, there was a significant analogue \times linking interaction, $F(1, 83) = 5.97$, $p = .02$, $\eta_p^2 = .07$ (see Figure 1). Among participants who were *not* asked to link to the analogue during the lesson, the predicted pattern was observed: Those who received the division analogue worksheet prior to the lesson made greater gains on the conceptual scale than did those who received the fraction analogue worksheet ($M_{\text{DnL}} = 25.5\%$ vs. $M_{\text{FnL}} = 10.0\%$ improvement), $\beta = 0.30$, $t(83) = 2.08$, $p = .04$. However, among participants who were asked to link to the analogue, there were no differences in gains between participants who linked to the division analogue and those who linked to the fraction analogue ($M_{\text{DL}} = 8.4\%$ vs. $M_{\text{FL}} = 10.0\%$), $\beta = 0.20$, $t(83) = 1.38$, *ns*. Furthermore, for participants who received the division analogue worksheet, those who were *not* asked to link improved more than did participants who were asked to link, $t(83) = 2.31$, $p = .02$, contrary to our prediction that linking would be beneficial. For participants who practiced with the fraction analogue, the simple effect of linking was not significant, $t(83) = 1.15$, *ns*.

This analysis also revealed a significant effect of gender, $F(1, 83) = 5.07$, $p = .03$, $\eta_p^2 = .06$, and a marginal effect of grade, $F(1, 83) = 2.95$, $p = .09$, $\eta_p^2 = .03$, on conceptual gains. On average, controlling for other variables, girls improved more on the conceptual items than did boys

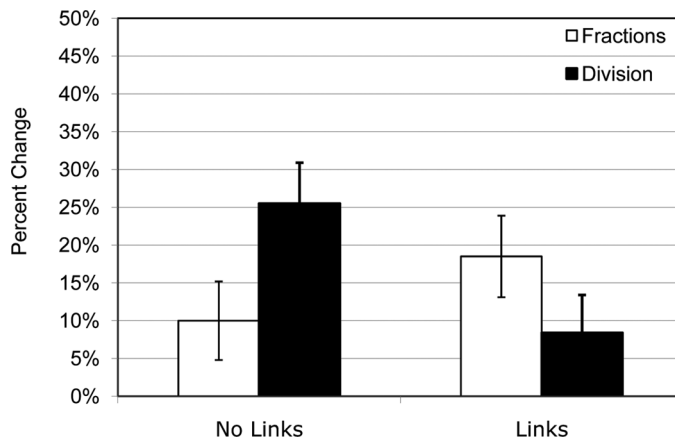


FIGURE 1 Students' conceptual change from pretest to posttest, by analogue and linking condition. The plotted group means have been adjusted for grade and gender.

($M_g = 23.7\%$ vs. $M_b = 11.3\%$) and older children improved more compared with younger children ($M_{6+} = 21.5\%$ vs. $M_6 = 16.1\%$ vs. $M_{5+} = 10.6\%$).

Procedural Transfer Performance

Because the two transfer equation items were scored dichotomously as incorrect (0) or correct (1), transfer performance was analyzed using a logit mixed model, as recommended by Jaeger (2008), using R's lmer function (lme4 library; Bates, Maechler, & Bolker, 2011). We fit a logit mixed model including analogue, linking, the analogue \times linking interaction, and pretest score as fixed factors and item as a random factor to the data. The fixed factor parameter estimates are shown in Table 2, with analogue, linking, and pretest mean-centered. There was no significant main effect of either analogue or linking, but there was a significant analogue \times linking interaction, Wald $Z = -2.44$, $p = .01$. The model that included the interaction term fit the data better than did a model without it, $\chi(1) = 5.80$, $p = .02$. To interpret this interaction, the analogue and linking factors were recoded such that their parameter estimates would reflect simple effects. Similar to the conceptual learning analyses presented earlier, the analogue mattered for

TABLE 2
Model Estimates for Procedural Transfer Items

Parameter	β Estimate	Std. error	<i>p</i> value
Intercept	-1.18	0.86	.17
Pretest	6.76	1.22	.00
Analogue	0.17	0.48	.73
Links	0.43	0.48	.38
Analogue \times Links	-2.38	0.98	.01

TABLE 3
Model Estimates for Picture Transfer Items

<i>Parameter</i>	<i>β Estimate</i>	<i>Std. error</i>	<i>p value</i>
Intercept	-1.07	0.20	.00
Pretest	1.32	0.83	.12
Analogue	-1.20	0.40	.00
Links	0.32	0.40	.41
Analogue × Links	0.01	0.79	.99

participants who were not asked to link, Wald $Z = 1.94$, $p = .05$, such that participants who received the division analogue were more likely to answer correctly than were those who received the fraction analogue ($P_{\text{DnL}} = 32.7\%$ vs. $P_{\text{FnL}} = 11.0\%$). However, there was no effect of analogue for students who were asked to link, Wald $Z = 1.53$, $p = .13$ ($P_{\text{DL}} = 18.7\%$ vs. $P_{\text{FL}} = 38.8\%$). Furthermore, for participants who received the fraction analogue before the lesson, those who were asked to link were more likely to answer correctly than were those who were not asked to link, Wald $Z = 2.35$, $p = .02$. For participants who received the division analogue before the lesson, there was no significant effect of linking, Wald $Z = 1.10$, $p = .27$.

Conceptual Transfer Performance

Far transfer on picture items. A logit mixed model was also used to analyze participants' responses on the two transfer picture items. We fit a model including analogue, linking, the analogue × linking interaction, and pretest score as fixed factors and item as a random factor to the data. The model estimates are shown in Table 3, with analogue, linking, and pretest mean-centered. The results from this model indicate a significant main effect of analogue, $B = -1.20$, Wald $Z = -3.03$, $p < .01$, but not linking, and no significant analogue × linking interaction. The model that included the analogue main-effect term fit the data better than did a model without it, $\chi(1) = 9.33$, $p = .01$. Contrary to our hypothesis, the results suggest that participants who received the fraction analogue were more likely to answer the picture items correctly than were those who received the division analogue ($P_{\text{Frac}} = 38.7\%$ vs. $P_{\text{Div}} = 16.0\%$ likely).²

Far transfer on story items. Very few participants were able to write transfer stories with any correct elements. Given the low rate of success and the small number of possible points on the story items (4), children's responses were analyzed as a count variable (ranging from 0 to 4) using a Poisson regression model including analogue, linking, the analogue × linking interaction, and pretest score as fixed factors. There was a significant main effect of linking, such that participants who were not asked to link during the lesson wrote correct story elements more often than did those who were asked to link to an analogue, $B = -2.04$, Wald $Z = -2.69$, $p < .01$. There was no significant effect of analogue and no analogue × linking interaction.

²This effect did not appear to be due to the division analogue supporting the "division makes smaller" misconception. We coded children's responses to the linking prompt in which children are asked to predict the size of the answer during the lesson. Of the 43 children in the linking conditions with codable videos, only 4 children (2 in each condition) incorrectly predicted size. In addition, children's verbal explanations did not express the "division makes smaller" misconception.

Quality of Conceptual Representations

Though participants in all conditions learned the “invert-and-multiply” procedure equally well, the simple effects of analogue type and linking on posttest conceptual change suggest that participants who received the whole number division analogue worksheet, but who were not prompted to make links, gained more conceptual knowledge about fraction division during the experiment than did participants in the three other groups (fraction analogue without linking, linking to division, and linking to fractions). To examine the possible causes of weak conceptual gains for most participants, we examined the errors students made in responding to posttest conceptual items. We focused on the quality of children’s representations on the story items.³

Coding. We coded the kinds of mathematical representations children used in their responses when asked to produce a fraction division representation. Specifically, we identified the operations (e.g., division or subtraction) and types of numbers (e.g., whole numbers or fractions) children used in the stories they generated. Coders determined the equation represented in a story by first identifying the “answer” to the story problem (e.g., the quotient), then the numbers used by the student (e.g., the dividend and divisor), and finally the operational structure fitting those numbers (e.g., division; see Appendix C for details). Two independent, trained coders agreed on 94% of reliability trials using this coding scheme on the story items.

Errors by condition. The equations students represented in their story problems were categorized into six types: *correct fraction division*, *whole number division*, *fraction multiplication*, *invert-and-multiply*, *other operations on fractions*, and *other operations on whole numbers*, and *no representation*. Coding definitions and examples of incorrect stories of each type are presented in Table 4. Nonmathematical stories or blank responses were categorized as *no representation*.

The percentage of posttest stories with *correct fraction division* representations is shown in Figure 2 by experimental condition. Participants who received the division analogue, but who were not asked to link, wrote correct stories most frequently, with 38% of all their stories written with a *fraction division* representation. Participants who linked to the division analogue wrote correct stories slightly less frequently (33% of stories), followed by those who practiced with the fraction analogue without linking (26%), and finally those who linked to the fraction analogue (24%). There was a trend such that participants who received the division analogue, across linking conditions, were more likely to produce correct representations on both story items (13 of 45 students) compared with those who received the fraction analogue, across linking conditions (6 of 44 students; $p = .07$, Fisher’s exact test, one-tailed).

Figure 2 also shows the percentage of posttest stories with *no representation*, by experimental condition. *No representation* was the most common story error in every condition. Participants who linked to the division analogue answered most frequently with *no representation*, depicting no mathematical representation in 33% of their opportunities to write stories. However, the

³Unfortunately, responses on the picture model items are difficult to interpret, as students’ reasoning for choosing a particular representation is sometimes unclear. For example, when asked to identify a picture to represent $5 \div \frac{1}{2}$, if a participant chooses a picture depicting $5 \times \frac{1}{2}$, it is unclear whether the picture was interpreted as fraction multiplication (seeing $5 \times \frac{1}{2}$) or whole-number division ($5 \div 2$). Thus, only the representations in students’ fraction division stories were used to examine error patterns.

TABLE 4
Types of Misrepresentation Errors for $7 \div 1/2$, and Percent of Stories With Each Misrepresentation Out of All Stories With a Misrepresentation (Number of Stories in Parentheses)

<i>Error type</i>	<i>Definition</i>	<i>Example</i>	<i>Fraction analogue</i>		<i>Division analogue</i>	
			<i>No links</i>	<i>Links</i>	<i>No links</i>	<i>Links</i>
$7 \times 1/2$	Story reflects <i>fraction multiplication</i>	Joe has 7 cakes. He wants to share them with a friend. If Joe gives his friend exactly $1/2$ of the cakes, how much will he give his friend?	14% (3)	19% (4)	39% (7)	56% (9)
$7 \div 2$	Story reflects <i>whole-number division</i>	I have 7 bananas. I divide my bananas into 2 different piles with the same amount of bananas in each. How many bananas are in each pile?	23% (5)	38% (8)	17% (3)	6% (1)
7×2	Story reflects <i>invert-and-multiply strategy</i>	There were 7 horses. Each of them ate 2 lb of hay each day. How many pounds of hay are eaten every day?	18% (4)	14% (3)	6% (1)	6% (1)
$7 - 2$	Story reflects <i>other operations on whole numbers</i> (usually subtraction)	Maria is selling 7 cakes and wants to sell 2 of them. How many will she have left if she sells 2 cakes?	23% (5)	14% (3)	22% (4)	13% (2)
$7 - 1/2$	Story reflects <i>fraction subtraction</i>	David is selling brownies for the farmers market. He has 7 brownies, and he ate $1/2$. How many does he have left?	23% (5)	14% (3)	17% (3)	19% (3)
Total number of stories with a misrepresentation			22	21	18	16

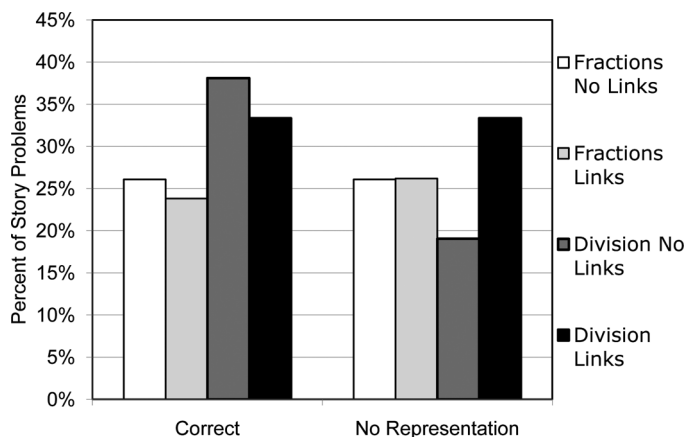


FIGURE 2 Percent of correct representations and no representations, by analogue and linking condition, on conceptual story problems at posttest.

frequency of responding with *no representation* did not differ significantly by condition, $\chi^2(3) = 2.36, ns$.

To gain insight into potential misconceptions developed by participants across conditions, we examined the frequency of particular kinds of misrepresentation in each condition. Table 4 presents the percentage of stories with a particular misrepresentation, out of all stories with a misrepresentation. When asked to write fraction division stories, participants who received the division analogue worksheet, regardless of linking, made *fraction multiplication* errors (16 of 34 errors) more often than did participants who received the fraction analogue worksheet (7 of 43 errors). Furthermore, of the participants who wrote incorrect stories, those who received the division analogue made fraction multiplication errors on both posttest story items (6 of 22 children) more often than did those who received the fraction analogue (1 of 27), Fisher's exact test, $p = .03$. These stories often had the flavor of whole-number division stories written with a fractional quantity, as in the fraction multiplication example presented in Table 4, Row 1.

In contrast, participants who received the fraction analogue worksheet, regardless of linking, wrote stories representing *whole number division* (13 of 43 errors) more often than did those who received the division analogue worksheet (4 of 34 errors). Indeed, only 1 participant who was asked to link to the division analogue wrote one whole number division story, whereas for participants who were asked to link to the fraction analogue, whole number division stories made up 38% of their misrepresentations (see Table 4, Row 2). Of the participants who wrote incorrect stories at posttest, 4 of the 27 participants who received the fraction analogue consistently made *whole number division* errors, but none of the 22 participants who received the division analogue did so, Fisher's exact test, $p = .08$.

There were no systematic differences by condition in the frequency of responding with other kinds of misrepresentations (see Table 4, Rows 3, 4, and 5).

Summary. Children's conceptual errors were diverse in all conditions; however, there were some trends. First, consider participants who received the division analogue worksheet. Those who were not prompted to link to the division analogue made the fewest errors overall—they

wrote the most *correct fraction division* stories at posttest and transfer. Those who were asked to link to the division analogue often produced *no representation* in their stories. When these children did depict incorrect representations, they tended to write stories about *fraction multiplication*, with the correct numbers but incorrect operation. It seemed that explicit prompts to link to the division analogue encouraged children to think about familiar whole number division situations (e.g., sharing with a friend) and represent them with a fraction, without properly adapting the division operational structure.

In contrast, participants who were asked to link to the fraction analogue frequently made *whole number division* errors at posttest. In this condition, it seemed that the explicit links between fraction division and other kinds of fraction problems highlighted the need to use a different operation, so children focused on writing stories that involved division representations. However, children's stories often did not include fractions. Participants who practiced with the fraction analogue, but who were not asked to link, made a variety of errors.

DISCUSSION

When learning in a new domain, children may spontaneously transfer their knowledge of previously studied problems to novel problems. According to a classic analogical transfer framework (Gentner, 1983), novices in a domain face several obstacles to positive transfer, including identifying the best source of prior knowledge (e.g., Novick, 1988; Novick & Holyoak, 1991) and adapting that knowledge in a way that supports new learning (e.g., Gick & Holyoak, 1980; Novick & Holyoak, 1991; Reed et al., 1985). In this study, we examined whether this framework could account for patterns of learning and transfer in an ecologically valid domain, fraction division, by exploring the relative utility of two analogue domains. Importantly, we sought to extend the literature on analogical transfer in mathematics by investigating the extent to which prior-knowledge analogues support conceptual learning, in addition to procedural learning. We expected that children's acquisition of knowledge about a new type of operational structure would be supported by positive transfer from a structurally similar domain but would be impeded by negative transfer from problems that had different operational structures. Furthermore, we expected that children would require opportunities for explicit linking to make the best use of the analogue.

Analogue Effects on Learning

Children in this study received explicit instruction about the invert-and-multiply procedure, and many children across all conditions learned the procedure and applied it to problems that were structurally identical to the ones they had encountered during the lesson. There were no effects of analogue or linking on children's gains in procedural knowledge of fraction division.

In contrast, children in this study were not explicitly instructed on the conceptual structure of division by fractions, and we found that analogue exposure and linking did affect their understanding of this conceptual structure. When children were not instructed to link, our analogical transfer hypothesis held true: Drawing on a structurally similar prior-knowledge domain proved to be better for new learning in the target domain compared with drawing on a surface-similar prior-knowledge domain.

Why was the division analogue effective in the absence of prompts to link? Children who practiced with the structurally similar analogue may have been implicitly oriented toward a division framework without direct attention to what they explicitly knew about whole number division. These findings are reminiscent of those of Day and Goldstone (2011), who also found performance gains when participants did not explicitly attend to similarities. Specifically, they found that after practicing in one novel domain, solvers showed better performance in a structurally similar but perceptually different novel domain, even if they did not report explicitly noticing similarities between the domains.

These findings seem surprising in light of Gentner's (1983) structure-mapping framework, in which analogical transfer is conceived as an explicit, controlled process in which mapping between elements is a critical part of analogical reasoning. However, recent connectionist accounts (Leech, Mareschal, & Cooper, 2008) provide an alternative perspective in which analogical reasoning is partially supported by a relational priming mechanism. In the priming account, exposure to a base domain primes specific relations in the target domain, without requiring an explicit mapping process. Together, these results suggest that learners need not explicitly notice the relevance of an analogue domain, nor must they explicitly map across domains, to gain conceptual understanding. Indeed, this may be one way in which analogical transfer of conceptual knowledge may differ from analogical transfer of procedures, which may require more explicit mapping (Gick & Holyoak, 1980; Novick & Holyoak, 1991; Reed et al., 1985). Further work will be needed to address this possibility.

A similar pattern of effects was observed on adaptation of procedures to items unlike the target problems, specifically those with more complex operands. Though many children learned the invert-and-multiply procedure with unit fractions, children who received the division analogue but who were not asked to explicitly link to it outperformed the other groups when applying this procedure to problems with more complex operands. Because this group was also highest in conceptual gains, this finding is consistent with research demonstrating links between high conceptual knowledge and successful adaptation of procedures for novel problems (Hecht & Vagi, 2011; Rittle-Johnson et al., 2001). These results suggest that children's conceptual knowledge did indeed support procedural adaptation.

Finally, though we found no effect of analogue on transfer story items, we found a negative effect of the whole number division analogue on transfer picture items, relative to the fraction analogue. In contrast to the verbal story items, visual representations of fraction division rely on identifying the correct fractions in the diagram as well as their correct relationship to one another. It may be that children's opportunity to draw fraction magnitudes in the fraction analogue conditions gave them an advantage in choosing the visual representation that included both operands. This finding highlights the multifaceted nature of conceptual knowledge and suggests that different analogues may highlight different facets of conceptual knowledge within a target domain.

Is Linking Unhelpful?

Our main finding—that a structural analogue can support some kinds of conceptual learning better than a surface analogue—did not hold in the conditions in which students were prompted to link. Furthermore, we found a negative effect of linking on one transfer conceptual scale. To

better understand these effects, we examined children's conceptual errors on story representations of fraction division. We found that children who were asked to link often wrote structurally inaccurate stories that suggested that they were concentrating on the *differences* between the prior-knowledge analogue to which they linked and the target domain. As a consequence, children who linked to different analogues made different types of errors. In general, those who compared to whole number division focused on writing stories that included fractions and those who compared to other operations on fractions focused on writing stories that represented the operational structure of division.

Based on our analysis of the story problems, it seems that instruction laid the foundation for the process of structural alignment (Gentner, 1983; Markman & Gentner, 1993). When comparing two problems, learners attempt an alignment of the problem structures, mapping across the relational similarities they can identify. In addition to facilitating analogical transfer, structural alignment also highlights relevant differences in the features of compared problems (Gentner & Gunn, 2001; Gentner & Markman, 1994). In our study, children who linked to the division analogue may have recognized the applicability of their whole number division knowledge, while also recognizing a need to use fractions. Thus, these children used their knowledge of familiar whole-number division contexts (e.g., equal sharing) without preserving the division relationship between quantities. In contrast, children who linked to fractions noticed the difference in operation (an alignable difference; see Gentner & Gunn, 2001), and they were more likely to use a familiar division structure. In both cases, children tended to focus on the differences between domains.

It appears that asking participants to link initiated an explicit structural alignment process and a potentially laborious, error-prone adaptation process. Our findings suggest that explicit linking is sometimes unhelpful for drawing on children's prior knowledge. However, other researchers have found positive effects of explicit comparison of similar problems (e.g., Gentner et al., 2003; Rittle-Johnson & Star, 2007; Star & Rittle-Johnson, 2009). In these studies, participants are typically asked to compare, or explain similarities and differences between, static isomorphic and highly similar problems. Our linking manipulation differs from this sort of comparison instruction in many ways. Instead of comparing a set of juxtaposed, very similar problems, we asked participants to think about a well-practiced body of knowledge (either whole number division or fraction addition and subtraction) and connect it to a novel type of problem. Children were free to make any connections that they noticed between the analogue and target domains. Novices may make any number of unhelpful links, which may impede their ability to focus on the most important connections across problems.

When drawing on prior knowledge, Thompson and Opfer (2010) found that directed linking better supports analogical transfer than does unguided linking. Thompson and Opfer had children compare smaller-magnitude number lines (i.e., 0 to 10) to larger-magnitude number lines (e.g., 1 to 1,000) to boost children's magnitude estimation of larger numbers. They found that an instructional condition in which children could make many irrelevant comparisons was less beneficial than a condition in which comparison was guided through progressive alignment of similar problems. In the guided condition, children were more likely to make relevant comparisons, and this resulted in more robust analogical transfer. These results suggest that children might show a greater benefit from analogical transfer from a structurally similar prior-knowledge domain if instruction guides children to make specific, relevant comparisons between highly similar problems.

Thompson and Opfer (2010) constrained children's comparisons by limiting the information that children needed to align at any given time, but there are many other possible ways to support

learners' comparisons in classroom settings. Richland and McDonough (2010) used a variety of cues to help undergraduate students perceive structural similarity and dissimilarity between math problems, including spatial alignment and comparative gesture, in addition to having the prior-knowledge analogue present during learning. They found that these cues enhanced learning, even after a 1-week delay, and in particular supported learners' abilities to inhibit transfer from surface-similar, but structurally dissimilar, problems. Together, these studies highlight the importance of drawing learners attention to key structural elements across sets of corresponding problems to best support knowledge transfer across problems.

Building on these comparison studies and our own work, one potentially fruitful direction for future research would be to examine the utility of specific links between domains, based on the particular facet of conceptual knowledge one hopes to boost. This study demonstrated the potential of both prior-knowledge domains to support different aspects of conceptual knowledge: Whole number division supported the operational structure of fraction division, and other fraction problems supported fraction magnitude representations. More targeted and structured comparisons between one analogue domain and another may better highlight key conceptual similarities between domains, while simultaneously avoiding misconceptions that are based on misleading similarities.

Implications for Educational Practice

The literature on analogical transfer largely focuses on transfer of knowledge in "artificial" tasks that have been constructed for use in laboratory contexts (e.g., Gick & Holyoak, 1987; Gentner et al., 2003; Novick, 1988; Novick & Holyoak, 1991). In this research, we have demonstrated with ecologically valid tasks that drawing on children's own prior knowledge of structurally similar domains can indeed support new conceptual learning, at least under some conditions. However, we have also demonstrated that different sources of prior knowledge highlight different conceptual facets of the target problem domain. When the instructional goal is to support students' understanding of the division structure underlying fraction division, lessons may be more useful when structurally similar analogues, such as whole-number division, are practiced first.

Our findings suggest that self-guided linking, in which students are asked to make links across problems without instructional support for noticing useful similarities and differences, is likely not the most beneficial way to implement analogical transfer in instruction. The degree of instructional support that teachers typically offer varies across cultures: American teachers frequently point out links between problems, but they implement substantially fewer instructional practices than do their Japanese and Chinese counterparts to support students' understanding of the relevant aspects of those links (i.e., gestures and spatial alignment; Richland, Zur, & Holyoak, 2007). Our data suggest that teachers and instructional materials should take care to support specific, useful comparisons by guiding attention to important structural features.

Conclusions

Analogical transfer is a mechanism by which we can draw on past experiences to boost our understanding of new problems. In this research, we demonstrated that a prior-knowledge

analogue can promote learning, not only in artificial laboratory tasks (as in most past research), but even when the analogue domain is an ecologically valid, highly practiced, and conceptually rich task (e.g., whole number division). Moreover, prior-knowledge analogues can be beneficial, not only for learning of procedures, but also for transfer of conceptual structure. In brief, practicing with structurally similar, familiar problems before a new lesson can support spontaneous, correct transfer of children's prior conceptual knowledge to a novel domain.

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APPENDIX A: ANALOGUE WORKSHEETS

Warm-Up Problems: Division

For Questions 1, 2, and 3, complete the equation.

1. $36 \div 6 =$

2. $40 \div 4 =$

3. $26 \div 2 =$

4. Consider the equation $20 \div 5 = 4$. Draw a picture that represents this equation.

5. How is $6 \div 2$ similar to $2 \div \frac{1}{4}$?

Warm-Up Problems: Fractions

For Questions 1, 2, and 3, complete the equation.

1. $\frac{1}{3} + \frac{1}{6} =$

2. $\frac{2}{5} + \frac{1}{4} =$

3. $\frac{5}{9} - \frac{2}{9} =$

4. Consider the fractions $\frac{1}{2}$ and $\frac{1}{4}$. Draw a picture that shows how they are related.
5. How is $6 - \frac{1}{2}$ similar to $2 \div \frac{1}{4}$?

APPENDIX B: LESSON SCRIPT AND WORKSHEET

Exp: Now we're going to go over a short lesson on dividing by fractions. Along the way, I'll ask you a couple of times to think about the warm-up problems you just did. Also, I'm going to ask you to use what you learn during this lesson to help you on the later problems, so you should make sure you understand the lesson. OK?

To begin, let's consider this story:

The Acme construction company is building a new road. The road will be 6 miles long. The Acme workers can build $\frac{1}{2}$ mile every day. How many days will it take to build all 6 miles?

How would you represent this with numbers and symbols? (*Allow student to write down representation.*)

- a. If student writes correct rep: Great job . . .
- b. If student writes completely incorrect phrase: Not quite. Actually . . .
- c. If student writes 6×2 : That calculation will get you the answer, but there is a better representation.

. . . this word problem can be represented as $6 \div \frac{1}{2}$. You need 6 miles of road, and you want to know how many times $\frac{1}{2}$ goes into 6.

Is there something about this problem that reminds you of your warm-up problems? How can you use those problems to think about this one? (*Allow student to answer verbally, referring to sheet of warm-up problems.*)

Keep thinking about your practice problems. Can they help you think about whether the answer will be bigger than 6 or smaller than 6?

Now I'm going to show you how to divide by a fraction like $\frac{1}{2}$. You'll see some examples and I'd like you to fill in the blanks as you follow along. To divide by a fraction, you can use the "invert-and-multiply" strategy. This strategy uses the idea that dividing by a number is the same as multiplying by its inverse. How does this relate to your warm-up problems? Good. Now I'm going to show you how to do the "invert-and-multiply" strategy. First you find the inverse of the divisor, which is the fraction you are dividing by. To find the inverse, you can "flip" a fraction. So (*point to first example problem*), the inverse of $\frac{1}{3}$ is $\frac{3}{1}$, which is the same as 3. Remember, any whole number can be represented as that number over 1. For example (*point to first example problem*), 3 represents three wholes or 3 over 1. We can also find the inverse of 3 by "flipping" 3 over 1 to get $\frac{1}{3}$. Here is another example (*point to second example problem*). The inverse of $\frac{1}{4}$ is 4 over 1. The inverse of 4 is 1 over 4. So, do you understand how to invert a fraction?

- a. Student replies "yes": Great. Can you tell me what the inverse of $\frac{1}{5}$ is? How about 5?
 - Correct answer: That's right. The inverse of $\frac{1}{5}$ is 5 and the inverse of 5 is $\frac{1}{5}$.
 - Incorrect answer: Go to "b"

- b. Student replies “no”: To invert a fraction, you switch the numerator and the denominator of a fraction. For example, the inverse of $1/5$ (*point to each number*) is $5/1$. $5/1$ is a way of representing 5 wholes, or just 5. Because 5 (*point to the 5*) is the same as 5 over 1, the inverse of 5 is 1 over 5, or $1/5$.

Remember the strategy to divide by a fraction is called the “invert-and-multiply” strategy. We’ve talked about how to invert a fraction, now we’ll talk about how to multiply two fractions. To multiply two fractions, you multiply the numerators of the fractions and then the denominator of the fractions. For example (*point to first multiplication problem*), to multiply $1/3$ by $1/5$, multiply 1×1 to get the numerator of your answer and 3×5 to get your denominator. 1×1 is 1 and 3×5 is 15, so $1/3 \times 1/5$ is $1/15$.

Can you multiply $2/5 \times 3/7$?

- Correct answer: That’s right. $2/5 \times 3/7$ is 6 over 35, or $6/35$.
- Wrong answer: That’s a good try. 2×3 is 6 and 5×7 is 35, so your answer is 6 over 35 or $6/35$.

Now, we’re going to put the invert-and-multiply together to solve the Acme Road problem, which we represented as 6 divided by $1/2$.

To solve $6 \div \frac{1}{2}$ find the inverse of $\frac{1}{2}$ and multiply it by 6. I’m going to show you the steps, and I’d like you to give me a reason for each step. (*Read each step and ask, “Why can I do that?”*) For a correct answer: That’s right. *Read reason.* Incorrect answer: Actually, I can do that because . . . *read reason.*

$6 \div \frac{1}{2} = 6 \times \frac{2}{1}$	Dividing by a fraction is the same as inverting and multiplying, and the inverse of $1/2$ is $2/1$.
$6 \times \frac{2}{1} = \frac{6}{1} \times \frac{2}{1}$	Another way we can write 6 or 6 wholes is $6/1$.
$\frac{6}{1} \times \frac{2}{1} = \frac{12}{1}$	To multiply two fractions, we multiply the numerators and the denominators, $6 \times 2 = 12$ and $1 \times 1 = 1$.
$\frac{12}{1} = 12$	Another way to write $12/1$, or 12 wholes, is 12.

So, $6 \div \frac{1}{2} = 12$. It will take the Acme construction company 12 days to build 6 miles of road. Twelve is bigger than 6. Does this seem reasonable? How was this problem like the kinds of warm-up problems you did?

Good. Now I’d like you to try a few problems on your own. These problems are similar to the ones I worked out for you in the middle of this page. The first step is the same, but the second two steps are a little different but very similar. For example, in this practice problem, I’ve simplified to a whole number before simplifying. For these problems, I’d like you to fill in the underlined blanks with expressions that are equivalent to the expression above it.

1.
 - a. Incorrect answer: Here you wrote (*student’s answer*), but the answer is 55, because $5/1$ times $11/1$ is 55.
 - b. Correct answer: That’s great, the answer is 55, because $5/1 \times 11/1$ is 55.
2.
 - a. Incorrect answer: Here you wrote (*student’s answer*), but the answer is 60, because $3/1 \times 20/1$ is 60.
 - b. Correct answer: That’s great, the answer is 60, because $3/1 \times 20/1$ is 60.

- 3.
- Incorrect answer: Here you wrote (*student's answer*), but the answer is 60, because $4/1 \times 15/1$ is 60.
 - Correct answer: That's great, the answer is 60, because $4/1 \times 15/1$ is 60.

APPENDIX C: STORY PROBLEM CODING

Equation Coding: equation represented, setup, and question

STEP 1

Figure out the ANSWER to the story problem the child has written. Your equation will need to be equal to this answer.

STEP 2

Figure out the equation.

- The equation should typically use the exact numbers that the student wrote. This can get hairy when students write out fractions (e.g., “one out of seven”). See the section about deciding between multiplication and division.
- Addition and subtraction will have no grouping structure (e.g., I have x and will have y more [or less]). Repeated subtraction still counts as subtraction, as long as there are no groups.

Deciding between multiplication and division

- Rate problems are generally multiplication by whole numbers, because there is no set end limit to what the larger number (the answer) will be.
 - Example: Can build x in y days, how long will it take to build z?
 - Example: One costs \$X, how much money will I make if I sell Y of them?
- If the answer will be more than the starting number AND it has a set total amount (the total is given), it is division by a fraction.
- If the answer will be less than either starting number, it is division by whole number OR multiplication by a fraction.
 - If there is a fraction in the problem, it is multiplication by a fraction.
 - If there are only whole numbers in the problem, it is division by a whole number

STEP 3

Figure out if the setup and question are correct.

1. Question is correct if the answer to the students' word problem is exactly the answer to the given equation.
2. Setup is correct if the story problem WOULD be correct IF the question was different.

Error Coding: Based on Equations

No representation: Story has no mathematical content, or no story at all.

Correct: Equation primarily involves division by a fraction.

Whole-number division: Equation primarily involves division by a whole number.

Fraction multiplication: Equation primarily involves multiplication by a fraction.

Invert-and-multiply: Equation involves the given dividend times the inverse of the given divisor.

Other operations on fractions: Equation primarily involves addition or subtraction by a fraction.

Other operations on whole numbers: Equation primarily involves addition or subtraction by a whole number.

- In simple equations with two operands and an operation, this coding will be straightforward, simply coded by the operator and nature of the operands.
- In more complex equations (e.g., $5 - (\frac{1}{7} \times 5)$ or $(5 \div 7) \times 10$), focus on the primary operation on the given first operand (5) and the nature of the number that is the argument of that operand.

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