The Quarterly Journal of Experimental Psychology

ISSN: 1747-0218 (Print) 1747-0226 (Online) Journal homepage: http://www.tandfonline.com/loi/pqje20

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To cite this article: Matthew J. Jiang, Jennifer L. Cooper & Martha W. Alibali (2014) Spatial factors influence arithmetic performance: The case of the minus sign, The Quarterly Journal of Experimental Psychology, 67:8, 1626-1642, DOI: 10.1080/17470218.2014.898669

To link to this article: https://doi.org/10.1080/17470218.2014.898669

Published online: 14 Apr 2014.

THE QUARTERLY JOURNAL OF EXPERIMENTAL PSYCHOLOGY, 2014
Vol. 67, No. 8, 1626–1642, http://dx.doi.org/10.1080/17470218.2014.898669

Spatial factors influence arithmetic performance: The case of the minus sign

Matthew J. Jiang[1], Jennifer L. Cooper[2], and Martha W. Alibali[3]

[1]Department of Psychology, Northwestern University, Evanston, IL, USA
[2]Wisconsin Center for Education Research, University of Wisconsin–Madison, Madison, WI, USA
[3]Department of Psychology, University of Wisconsin–Madison, Madison, WI, USA

Spatial features of mathematical equations may influence how people solve and interpret those equations. This study examined whether manipulations of spatial features affected how participants solved and interpreted equations involving the minus sign. Undergraduate participants ($N = 91$) solved multioperation arithmetic equations involving addition, subtraction, and multiplication (e.g. $25 - 3 + 2 \times 5 =$ __). We varied the spacing of the final three operands and the position of the first operator relative to the adjacent operands. Participants also generated a story problem to correspond with a given equation. We evaluated the procedures that participants used in solving the equations by analysing both their solutions and their written work. Both close spacing of the final three operands and position of the first operation sign influenced the procedures that participants used. Both of the spatial manipulations also influenced participants' interpretations of the conceptual structure of the equations, as revealed in the story problems that they generated. These results have implications for understanding how people process mathematical symbols and for mathematics education.

Keywords: Subtraction; Arithmetic; Spatial proximity; Mathematical symbols; Mathematical operations.

Mathematical notations are based on conventions. For example, the rules for order of operations are conventions that specify the hierarchy of steps through which expressions can be simplified. Some conventions are specified by spatial features of the notation. For example, a superscript delineates an exponent rather than a regular integer. In this research, we investigate spatial features of equations, with a focus on how people solve and interpret multioperation equations such as $25 - 3 + 2 \times 5 =$ __.

SPATIAL FACTORS IN SOLVING AND INTERPRETING EQUATIONS

Although some spatial features affect the meaning of mathematical expressions (e.g. 3/4 does not mean the same thing as 4/3), others have no influence on how expressions should be interpreted. For instance, the spacing between operators and numerals does not influence how the operations should be interpreted—the expressions "3 + 2" and "3+ 2" are formally identical to

Correspondence should be addressed to Matthew Jiang, Department of Psychology, Northwestern University, 2029 Sheridan Road, Evanston, IL 60208, USA. E-mail: jiang@u.northwestern.edu

Portions of these data were presented at the 2013 Biennial Meeting of the Society for Research in Child Development (Seattle, Washington). In addition, portions of these data were presented at the 2013 Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education (Chicago, Illinois) and were described in the proceedings of that meeting.

We are grateful to Xinya Chuong and Kali Schulte for assistance with coding.

Partial funding for this research came from a Rosevear Research Scholarship from the Department of Psychology at the University of Wisconsin to M.J.J.

"3 + 2". For interpreting operations, the order of the operands and the operators is critical, not the width of their spacing. When solving mathematical expressions, knowing the rules for order of operations (e.g. solving multiplication before addition) enables calculation of the formally correct solution regardless of spacing.

Nevertheless, recent findings indicate that the spacing of numerals and operations influences how people solve and interpret equations (e.g. Kirshner, 1989; Landy & Goldstone, 2007b, 2010). One line of research on spacing focuses on the rules for order of operations. Kirshner (1989) and Landy and Goldstone (2007a, 2007b, 2010) reported that people interpret operations with narrower horizontal spacing as having precedence, even when it conflicts with the formal rules for order of operations. Manipulating spacing so that it conforms to the conventional rules for order of operations (e.g. spacing multiplication operations more narrowly than addition operations) improves judgements of expression equality (Landy & Goldstone, 2007b), decreases errors in equation solving (Landy & Goldstone, 2007b, 2010), and decreases reaction time (Landy & Goldstone, 2007b, 2010), relative to spacing that is inconsistent with the rules for order of operations (e.g. spacing addition operations more narrowly than multiplication operations).

These studies indicate that metric spatial properties can affect reaction time and accuracy. Although "3+2 × 5" and "3 + 2×5" share the same ordinal properties (i.e. the ordering of the symbols is strictly identical), the metric properties are different, in that the amount of horizontal spacing between these symbols varies. The key point from these studies is that even for rules that rely solely on ordinal spatial properties (such as order of operations), metric spatial properties that are not part of the rule system (such as horizontal spacing) can affect processing.

Taken together, these studies indicate that participants do not process rules for order of operations solely in terms of what the symbols mean. If they did, these horizontal manipulations would have no effect on performance. Instead, the data suggest that spatial features of equations influence the steps people take in solving the equations (and how quickly they take those steps). In addition, the data suggest that spatial features influence people's interpretations of the conceptual structure of the equations—for example, in their judgements of equation equality (Landy & Goldstone, 2007b). Thus, spatial factors influence both procedural and conceptual aspects of people's equation-solving performance.

This body of work is compatible with other research that emphasizes the integrated nature of perceptual, procedural, and conceptual factors in mathematics learning and problem solving (e.g. Goldstone, Landy, & Son, 2010; McNeil & Alibali, 2005; Rittle-Johnson & Koedinger, 2005). Ties between perceptual encoding and procedural knowledge are well established. People who encode mathematics problems accurately are more likely to use correct procedures in solving the problems (e.g. Alibali, Phillips, & Fischer, 2009; Booth & Davenport, 2013). Moreover, the specific features that people encode are closely tied to the procedures they use (e.g. McNeil & Alibali, 2004). In light of this past work, it is reasonable to expect that spatial features that affect perceptual grouping would also affect the problem-solving procedures that people use and the errors they make.

There are also close ties between perceptual processes and conceptual knowledge (Goldstone & Barsalou, 1998). Although there continues to be debate about the perceptual bases of abstract concepts (see Barsalou, 2010, for a review), there is accruing evidence that perceptual processes play an important role in mathematics learning (e.g. Goldstone et al., 2010; Kellman, Massey, & Son, 2010). Moreover, because conceptual and procedural knowledge are interconnected (e.g. Rittle-Johnson, Siegler, & Alibali, 2001), it seems possible that perceptual processes could affect conceptual knowledge via their effects on people's use of procedures (Alibali, 2005). Thus, it seems likely that spatial features that affect perceptual grouping could also influence the conceptual knowledge that people activate and use.

UNDERSTANDING AND USING THE MINUS SIGN

The bulk of past work on spatial factors in arithmetic has focused on addition and multiplication (e.g. Landy & Goldstone, 2007a, 2007b, 2010; Landy, Jones, & Goldstone, 2008). In the present study, we investigate whether spatial manipulations influence participants' performance on equations involving the minus sign. Several lines of research suggest that minus signs present people with particular challenges, making subtraction an important target of study. First, subtraction problems are solved more slowly than corresponding addition problems, even in adults (e.g. Das, LeFevre, & Penner-Wilger, 2010). Second, middle-school students often display poor conceptual understanding of subtraction (Dixon, Deets, & Bangert, 2001). Third, middle-school students sometimes erroneously "detach" the subtraction sign from the following operand when solving multioperation arithmetic problems (Herscovics & Linchevski, 1994). For example, Herscovics and Linchevski (1994) reported that, when solving "$4 + n - 2 + 5$", over two thirds of their seventh-grade participants initially calculated $2 + 5$ (yielding the expression $4 + n - 7$). Herscovics and Linchevski also found similar detachment errors in problems without unknown variables.

Understanding students' difficulties with the minus sign is particularly important because the minus sign is used not only to denote subtraction, but also to denote negative numbers. Thus, unlike other operators, the minus sign is used in a broader range of ways as students advance to higher level mathematics (Vlassis, 2004). Students who have difficulty understanding the diverse ways in which the minus sign can be used may also encounter additional challenges at the transition to algebra (Cangelosi, Madrid, Cooper, Olson, & Hartter, 2013). It therefore seems worthwhile to investigate how spatial features of equations may play a role in supporting or hindering people's use and interpretation of the minus sign.

In this work, we extend past findings about how spatial features of equations influence solvers' application of rules for order of operations to subtraction. Toward this aim, we presented participants with two types of multioperation arithmetic equations that involved addition, subtraction, and multiplication ($a + b - c \times d = \_\_$ and $a - b + c \times d = \_\_$). We examined how variations in spacing affected both the procedures that participants used in solving the equations and their conceptual interpretations of the equations.

THE PRESENT STUDY

In the present study, we addressed two primary questions about how spacing affects solvers' use and interpretation of equations involving minus signs.

First, does spacing affect the procedures that people use to solve equations that involve minus signs? To address this question, we looked both at the final solutions that participants provided and at participants' written work, which in many cases showed their procedural steps. Past studies of spatial manipulations of equations have focused primarily on accuracy and reaction time; we believe that examining written work can provide unique insights into participants' procedure use. We evaluated whether participants' solution processes adhered to the rules for order of operations, and we evaluated the nature of their errors.

Second, does spacing influence people's conceptual interpretations of equations involving minus signs? To address this issue, we asked participants to generate a story problem that corresponded with a given equation. From their responses, we assessed whether the spatial manipulations affected their interpretations of the equation. We also asked participants to solve a story problem, so that we could evaluate whether the spatial manipulations (which they had seen in prior problems) influenced their interpretations of a story problem that did not itself incorporate those spatial manipulations. This research is the first to utilize story problems in examining the effects of spatial manipulations on interpreting and solving equations.

Table 1. *Variations in operand spacing and first operator position for a subtraction-first equation*

	First operator position		
Operand spacing	*Left shift*	*No shift*	*Right shift*
Evenly spaced	25 – 3 + 2 × 5 = __	25 – 3 + 2 × 5 = __	25 –3 + 2 × 5 = __
Closely spaced	25 – 3+2×5 = __	25 – 3+2×5 = __	25 –3+2×5 = __

Previous studies of operation spacing have utilized manipulations that group operations and operands more narrowly or more widely (e.g. 4+5 vs. 4 + 5). In this research, we extend past findings to study shifts in operator position in which the spacing of the numbers themselves is not altered. That is, we examine lateral shifts in operator position, with the spacing of the adjacent numbers held constant. This research is the first to examine the effects of this type of spatial manipulation.

We varied these two aspects of spacing—*operand spacing* and *first operator position*—in a factorial design. Table 1 displays the two factors and the specific levels we used. We manipulated operand spacing by removing the spaces between the operands and operators for the last three terms in the equations. This manipulation is similar to that used by Landy and Goldstone (2007b). We manipulated first operator position by shifting the first operator horizontally, so that it was closer to the numeral that preceded it, or closer to the numeral that followed it. For both factors, the variations were subtle, consisting of only one or two spaces in 12-point font. Although we could have tested many other spatial manipulations, we selected these specific ones because we expected that they would influence participants' error patterns in predictable ways. We examined two primary types of errors in participants' equation solving: detachment errors and left-to-right errors. We turn next to a description of these errors.

As noted above, we utilized two types of equations, which we termed addition-first equations ($a + b - c \times d = \_\_$) and subtraction-first equations ($a - b + c \times d = \_\_$). In problems of the form $a - b + c \times d$, in which subtraction precedes addition, participants could erroneously "detach" the minus sign (like the participants in Herscovics & Linchevski, 1994) and solve from right to left, starting with the multiplication operation (as would be correct according to the rules for order of operations) and working backwards (violating the rules for order of operations). This solution procedure yields what we call a *detachment* error, as shown in the following example:

$$25 - 10 + 2 \times 3 = \_\_$$
$$25 - 10 + 6 = \_\_$$
$$25 - 16 = 9 \text{ [incorrect]}$$

In this error, the minus sign is detached from the subtrahend in the intermediate step: After multiplying 2×3, the solver ignores the minus sign and adds 6 to 10. The solver then subtracts the resulting quantity, 16, from 25, yielding the erroneous solution of 9. Thus, the solver performs "$25 - [10 + (2 \times 3)] = \_\_$" instead of the presented equation, which is "$25 - 10 + (2 \times 3) = \_\_$" (parentheses added).

Participants could also produce *left-to-right* errors, which involve solving the equations from left to right, ignoring the rules for order of operations altogether. Left-to-right errors could occur on addition-first and subtraction-first equations, but detachment errors could occur only on subtraction-first equations. Therefore, in the following sections, we describe our hypotheses with respect to subtraction-first equations. We used the addition-first equations to obtain a "clean" estimate of participants' tendency to make left-to-right errors, because addition-first equations are not subject to detachment errors.

Hypothesized effects of the spatial manipulations on equation-solving procedures

We hypothesized that close spacing of the last three operators (e.g. 25 − 3+2×5) would lead participants to perceptually group these terms due to their proximity, as suggested by the Gestalt principles of visual perception (e.g. Wertheimer, 1923/1938) and by previous research (e.g. Landy & Goldstone, 2010). We further hypothesized that the resulting perceptual group would lead participants to make detachment errors, as described above.

We expected that participants would start with the multiplication operation inside the perceptual group and then solve the addition operation next, because it is part of the same perceptual group. Although this violates the rules for order of operations, we expected the narrower spacing around the plus sign than around the minus sign to lead participants to infer that the addition operation should take precedence over the subtraction operation (Landy & Goldstone, 2010). Participants would solve the subtraction operation last, because when carrying out the addition operation, they detached the operand from the minus sign, since the operand, but not the operator, was part of the perceptual group.

We also hypothesized that shifts in *first operator position* would lead participants to form different perceptual groups, as suggested by the Gestalt principles of visual perception (e.g. Wertheimer, 1923/1938). For example, in the expression "25 − 3 + 2 × 5", if the minus sign (the first operator) is shifted slightly to the left (e.g. 25 − 3 + 2 × 5), participants should be more likely to perceptually group the remaining symbols, "3 + 2 × 5". Like the perceptual group created by close spacing of the last three terms, this perceptual group might also encourage participants to "detach" the minus sign, leading participants to solve the multiplication operation first and then solve the addition operation before the subtraction operation.

In contrast, when the minus sign is shifted to the right, participants should be more likely to group the minus sign with the subsequent numeral and therefore incorporate the subtraction operation as part of the perceptual group (essentially making the entire equation a single perceptual group). As a result, participants should solve the multiplication operation first and then solve the subtraction and addition operations from left to right (i.e. first 25 − 3, yielding 22, and then 22 + 10, for the correct solution of 32).

Alternatively, when the minus sign is shifted to the right, solvers might "bind" the minus sign to the 3 as if it were a negative sign, essentially treating 25 −3 as 25 + (−3). Such binding is also a sort of ordering rule; the ambiguity of the minus sign, along with the spatial manipulations, might afford this sort of interpretation. If participants follow this interpretation, they should solve the multiplication operation first, then they should add the negative operand to that product (i.e. −3 + 10, yielding 7), and then they should perform the remaining addition operation (25 + 7, for the correct solution of 32). For present purposes, the key point is that in either case (i.e. if participants interpret subtraction as part of the perceptual group or if they bind the minus sign to the 3 as a negative sign), the right-shifted minus sign should promote correct solution procedures.

Thus, we expected both operand spacing and the minus sign position to influence participants' production of detachment errors. Specifically, we expected that, for equations of the form $a -\ b + c \times d =$ __, manipulations leading to perceptual grouping of the last three terms—such as close spacing of those terms and/or shifting the minus sign to the left—would lead to more detachment errors, in which participants perform addition before subtraction. The close spacing of the last three terms makes the addition operation part of a perceptual group at the end of the equation, and the left shift of the minus sign takes the subtraction operation out of that perceptual group. Therefore, both of these manipulations should promote detachment errors.

Finally, we also anticipated that manipulations leading to perceptual grouping of the last three terms might decrease left-to-right errors, in which participants simply perform the operations from left to right, in the order presented, ignoring the rules for order of operations. In all of the equations we used (both addition-first and subtraction-first), multiplication was the final operation. We hypothesized that perceptual groups that highlight the last two operations should help highlight the precedence of multiplication and thereby prevent participants from simply performing the given operations from left to right.

Hypothesized effects of the spatial manipulations on conceptual interpretations of equations

We also investigated how the spatial manipulations influenced participants' interpretations of the *conceptual structure* of the equations. We wished to know whether the effects of the spatial manipulations were limited to participants' equation-solving procedures at the moment of solving, or whether they went further, to influence their interpretations of the conceptual structure of the equations. To address this question, we asked participants to generate a story problem to correspond with a given equation. The story problem generation task has been used as a measure of conceptual understanding in previous research (Alibali, Brown, Stephens, Kao, & Nathan, 2009; Ma, 1999; Sidney & Alibali, 2013), but it has not yet been used with spatial manipulations of equation features. To generate a story problem based on an equation, the participant must interpret the structure of the equation and translate it into a meaningful story, in which information about mathematical relations is expressed via the semantic relations in the story.

We examined whether participant-generated story problems reflected an interpretation of the original problem that involved a detachment error [i.e. the problem was interpreted as $a - (b + c \times d) = \_\_$ rather than $a - b + c \times d = \_\_$]. We hypothesized that spatial manipulations that promote perceptual grouping of the last three terms would lead participants to generate story problems that manifested detachment errors, similar to the effects of those manipulations on equation-solving procedures.

Finally, we also asked participants to solve a story problem, which, by its nature, does not include spatial cues to the perceptual grouping of the last three terms. We expected that the supportive context of the story might shield participants from making the detachment and left-to-right errors that we expected them to make when presented with equations that involved spatial manipulations.

Method

Participants

Undergraduate students ($N = 91$, 52% male; $M_{\text{age}} = 19;5$; range 18;6 to 21;1) at a large Midwestern university participated for extra credit in introductory psychology. Sixteen additional participants were excluded due to a photocopying malfunction that added additional markings to the stimuli pages.

Design

The study utilized a 2 (operand spacing: evenly spaced or closely spaced) × 3 (first operator position: left shift, no shift, or right shift) between-subjects design, yielding a total of six conditions.

Examples of the stimuli are presented in Table 1. In the *evenly spaced* condition with *no shift*, there were two spaces between each operand and the adjacent operator and two spaces around the equal sign. In the *closely spaced* condition with *no shift*, we removed all of the spaces between the last three operands and their two operators. Thus, in the *no-shift* conditions, there was equal spacing (two spaces) between the first operand and first operator and between the operator and the second operand. The *left-shift* conditions were created by reducing the number of spaces between the first operand and first operator by one and increasing the number of spaces between the first operator and second operand by one. The opposite was done for the *right-shift* conditions.

We also varied the order in which the operations were presented in the equations. In *subtraction-first* problems, the subtraction operation was first, followed by addition and multiplication (e.g. $25 - 3 + 2 \times 5 = \_\_$). In *addition-first* problems, the addition operation was first, followed by subtraction and multiplication (e.g. $25 + 8 - 2 \times 3 =$ __). In all problems, multiplication was the final operation.

Materials

Participants received arithmetic equations (10 subtraction-first problems and eight addition-first problems) with manipulated spatial features. All participants solved the same set of 18 randomly generated arithmetic problems, presented in the same order (see Appendix A). The operands in the set of equations were randomly generated, with the constraint that the operands were positive integers, as were all correct, detachment error, and left-to-right error solutions. The operands were all single- or double-digit numbers.

Problems were presented in packets the size of a half sheet of A4 paper (21 cm × 14.85 cm) in size 12 Calibri font. On each page, there were two maths problems, separated by a filler vocabulary problem. Items were numbered at the left margin, and the equations were centred. The packets were constructed such that a page with subtraction-first problems was at the beginning, and those pages then alternated with pages with the addition-first problems. Problem order was not manipulated.

After the equation pages, participants were asked to generate a story problem to correspond with a given subtraction-first equation. To help them understand the task, they were provided an example of a simple equation ($7 + 5 =$ __) and a corresponding story problem ("Jane was on Lake Mendota fishing with her mother. She caught seven fish and her mother caught five. How many did they catch altogether?").

The final page of the packet contained the following story problem:

> Jake has a total of 45 dollars right now. He plans to pay 12 dollars for lunch today. Jake will also babysit for two hours this evening and earn 7 dollars an hour. How much money will he have by tomorrow morning?

This story problem was designed to correspond to a subtraction-first equation. Participants were asked to write the corresponding equation and to solve the story problem.

Procedure

Participants completed the packets individually. Up to five students were tested during each experimental session; each completed his or her own packet. Each participant was given up to 30 minutes to complete the packet of maths problems and vocabulary filler items; no participant required more than the allotted time.

Coding

Equation solutions. Solutions to equations were coded into four categories: correct, detachment error, left-to-right error, or other error. Solutions were coded as *correct* only if they were exactly correct (e.g. $25 - 5 + 2 \times 3 = 26$). Solutions were coded as *detachment errors* if they were consistent with participants having performed the addition in the second step, ignoring the minus sign [essentially solving the equation $a - (b + c \times d) =$ __, e.g. $25 - (5 + 2 \times 3) = 14$]. Solutions were coded as *left-to-right errors* if they were consistent with participants having performed the operations from left to right, without regard for the rules for order of operations [e.g. $(25 - 5 + 2) \times 3 = 66$]. All incorrect responses that did not fit into one of the aforementioned categories were considered *other errors* (6% total); these were a mixture of arithmetic errors, operator substitution, and other or unidentifiable error patterns.

Written solution procedures

Participants' written responses to the equations were also coded based on the work that they showed. We used their intermediate steps to identify the order in which the operations were performed. Details and examples are presented in Table 2.

As seen in the table, procedures for the subtraction-first equations ($a - b + c \times d =$ __) were coded into four primary categories: (a) *multiply-subtract*, in which participants solved the multiplication operation first, then the subtraction operation, and finally the addition operation; (b) *multiply-add negative*, in which participants solved the multiplication operation first, then added the second operand as a negative integer; (c) *multiply-detach*, in which participants solved

Table 2. *Solution procedure codes, examples, and percentage of participants in each condition who used each procedure at least once for each problem type, scored from their written work*

		Condition					
		Evenly spaced			Closely spaced		
Procedure	Example	Left shift	No shift	Right shift	Left shift	No shift	Right shift
Processes on subtraction-first problems							
Multiply-subtract	$25-3+2\times5$ $25-3+10$ $22+10$ 32	28	39	17	20	57	64
Multiply-add negative	$25-3+2\times5$ $25-3+10$ $25+(-3+10)$ $25+7$ 32	6	0	17	7	7	14
Multiply-detach	$25-3+2\times5$ $25-3+10$ $25-(3+10)$ $25-13$ 12	11	6	0	33	57	0
Left-to-right	$25-3+2\times5$ $22+2\times5$ 24×5 120	0	0	0	17	6	8
Never showed any of the above patterns	Varied	44	56	58	47	36	36
Processes on addition-first problems							
Multiply-add	$25+8-2\times3$ $25+8-6$ $33-6$ 27	22	33	17	33	50	50
Multiply-subtract	$25+8-2\times3$ $25+8-6$ $25+2$ 27	6	0	0	40	43	7
Left-to-right	$25+8-2\times3$ $33-2\times3$ 31×3 93	22	6	8	0	0	0
Never showed any of the above patterns	Varied	50	61	75	33	36	50

Note: Despite instructions to show their work, many participants did not do so. The category "never showed any of the above patterns" includes participants who never provided any written work ($n = 11$), as well as participants who never provided work that fitted into one of the listed categories. (They sometimes showed work for only one or two operations, or work that used incorrect operations, or work that combined multiple operations into a single step so the order of steps could not be discerned.)

the multiplication operation first and then added the second operand, detaching the minus sign; and (d) *left-to-right*, in which participants solved from left to right, solving subtraction first, then addition, then multiplication. Procedures that did not fall into one of these categories were coded as *other* and were not analysed further.

Procedures for the addition-first equations ($a + b - c \times d =$ __) were coded into three main categories, again based on the order in which the operations were performed: (a) *multiply-add*, in which participants solved multiplication first, then addition, and then subtraction; (b) *multiply-subtract*, in which participants solved multiplication first, then subtraction, then addition; and (c) *left-to-right*, in which participants solved from left to right, solving subtraction first, then addition, then multiplication. Note that in the addition-first problems, the detachment error cannot be distinguished, because detaching the plus sign does not affect the computation. Procedures that did not fall into one of these categories were coded as *other* and were not analysed further.

Participant-generated story problems. The story problems that participants generated were coded by attempting to solve them and identifying the equation that they represented. The coding categories used for equation solving (correct, detachment errors, left-to-right errors, and general errors) were also applied to the equations derived from the participant-generated story problems. As with the equation-solving items, the category of *other* errors included arithmetic mistakes in addition to other structural errors or unclear responses. When possible, ambiguous wording was clarified through any additional work the participant showed.

The story problems were initially coded by one of the authors (J.C.), and then all were independently coded by a research assistant. Agreement between the two coders was 96.5%. For data analyses, the author's original codes were used.

Story problem solutions. Solutions to the story problem were coded into the same categories as solutions to the equations.

Results

Did the spatial manipulations influence the likelihood that participants made detachment errors in solving equations?

We first asked whether our manipulations of spatial features of the equations (i.e. operand spacing and first operator position) influenced whether participants produced detachment errors. These errors could be identified only on subtraction-first problems, so our analysis focused on those problems. To address this question, we analysed whether or not participants ever produced a detachment error using logistic regression, with first operator position and operand spacing as fixed effects. Figure 1, left panel, presents the proportion of participants who produced at least one detachment error by condition. Among the participants who made at least one detachment error ($n = 22$), an average of 5.7 ($SE = 0.78$, $Mdn = 6.5$) of their 10 responses were detachment errors. The distribution of the number of detachment errors made by each participant is shown for each condition in Appendix B.

The best fitting model included main effects of both operand spacing and first operator position. A greater proportion of participants provided solutions indicating detachment errors in the closely spaced conditions than in the evenly spaced conditions, yielding a main effect of operand spacing, $z = 3.71$, $p < .001$. A greater proportion of participants also provided solutions indicating detachment errors in the left-shift condition than in the right-shift condition, $z = 2.36$, $p = .02$, and in the no-shift condition than in the right-shift condition, $z = 2.46$, $p = .01$.

To further test the significance of each main effect, we compared a model that included that effect with a nested model that did not include that effect. The model that included operand spacing provided a better fit to the data than the model that did not include operand spacing, change in deviance $= 17.51$, $p < .001$. The odds that participants produced a detachment error were 10.61 times greater in the closely spaced condition than in the evenly spaced condition, 95% confidence interval (CI) [3.31, 42.16]. Likewise, the model that

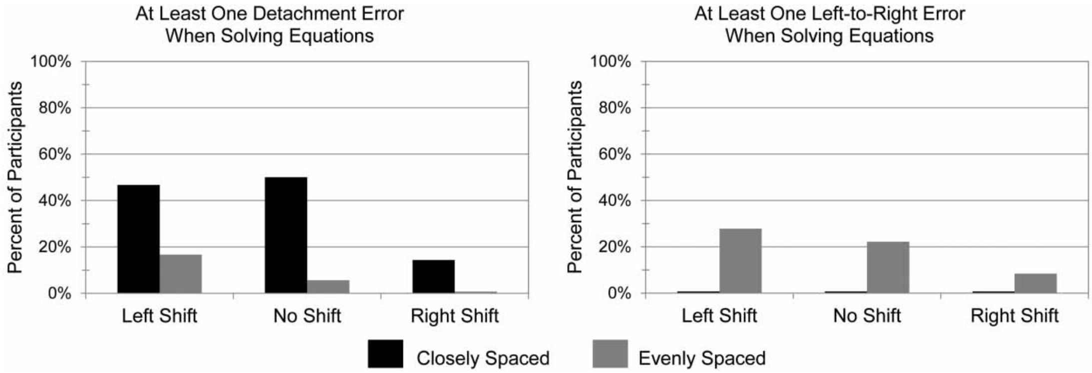

Figure 1. *(Left) Percentage of participants who made detachment errors on the subtraction-first equations, by condition. (Right) Percentage of participants who made left-to-right errors, by condition. Subtraction-first and addition-first problems are combined. Note that no participants made left-to-right errors in the closely spaced condition (i.e. all bars for that condition are 0).*

included first operator position provided a better fit to the data than the model that did not include first operator position, change in deviance = 9.17, $p = .01$. The odds that participants produced a detachment error were 7.95 times greater in the left-shift condition than in the right-shift condition, 95% CI [1.65, 59.90], and 8.84 times greater in the no-shift condition than in the right-shift condition, 95% CI [1.82, 67.32].

We also asked whether the likelihood of participants ever making a detachment error depended on the combination of first operator position and operand spacing. A model that included the interaction between first operator position and operand spacing did not provide a better fit than the model without the interaction, change in deviance = 2.43, $p = .30$. Thus, there was no evidence that the likelihood of participants ever making a detachment error depended on the interaction of first operator position and operand spacing.

Did the spatial manipulations influence the likelihood of left-to-right errors when solving equations?

We next asked whether our manipulations of spatial features of the equations influenced whether participants produced left-to-right errors. These errors could be identified on both subtraction-first and addition-first problems, so our analysis includes both problem types. Ten participants made at least one left-to-right error, and among these participants, an average of 9.9 ($SE = 2.4$, $Mdn = 13.5$) of their 18 responses included left-to-right errors.

As seen in Figure 1, right panel, no participants in the closely spaced conditions ever produced left-to-right errors—all of the participants who produced left-to-right errors were in the evenly spaced conditions. The complete absence of left-to-right errors in the closely spaced conditions suggests that the close spacing of the final three operands did indeed protect participants from making left-to-right errors. Five participants in the left-shift condition produced left-to-right errors, four in the no-shift conditions, and one in the right-shift condition.

Given the complete absence of left-to-right errors and the resulting lack of variability in the closely spaced conditions, we could not test the effects of the spatial manipulations using logistic regression; models failed to converge or coefficients had inappropriately large standard errors. Therefore, we utilized Fisher's exact tests to compare the proportion of participants who produced left-to-right errors in the closely spaced and evenly spaced conditions. For both subtraction-first and addition-first equations, a greater proportion of participants produced left-to-right errors in the evenly spaced condition than in the closely spaced condition, both ps = .0013, Fisher's exact (two-tailed).

Did participants' written work reveal how they grouped the operands in solving the equations?

To better understand the effects of our manipulations, we examined how participants grouped the operands when solving the equations. For subtraction-first problems, two possibilities were of special interest (see Table 2). First, participants could use the *multiply-subtract* procedure, solving multiplication first, then subtraction, and then addition (e.g. for the problem $25 - 3 + 2 \times 5 = \_\_$, they could first solve $2 \times 5 = 10$, then $25 - 3 = 22$, and then $22 + 10 = 32$). Alternatively, they could use the *multiply-add negative* procedure, solving multiplication first, then treating the second operand as a negative integer and adding it, and then solving the first addition (e.g. for the problem $25 - 3 + 2 \times 5 = \_\_$ they could first solve $2 \times 5 = 10$, then $-3 + 10 = 7$, then $25 + 7 = 32$).

As noted in the Method section, some participants did not provide any written work ($n = 11$, 12% of the sample), but many participants did so. Table 2 displays the percentage of participants in each condition who displayed each procedure at least once in their written work for addition-first and subtraction-first problems. Note that each participant could fall into more than one category depending on the varying procedures they used.

As seen in Table 2, for the subtraction-first equations, many participants showed written work that revealed that they performed subtraction before addition (i.e. the multiply-subtract procedure, line 1 of Table 2). A smaller number of participants treated the second operand as a negative integer and performed the addition of the negative as the second step, which also ultimately led to a correct solution (i.e. the multiply-add negative procedure, line 2 of Table 2). These descriptive results indicated that both approaches for arriving at correct solutions did occur. Future studies that utilize other designs (e.g. talk-aloud protocols) will be needed to investigate the factors that influence participants' use of these distinct procedures.

The data patterns in Table 2 align with the findings from the quantitative analysis of students' solutions. For subtraction-first equations, more participants in the closely spaced conditions than in the evenly spaced conditions displayed written work that reflected the multiply-detach procedure, and fewer participants in the right-shift condition than in the left-shift or no-shift conditions did so.

For the addition-first equations, many participants in the closely spaced conditions displayed written work that revealed a *multiply-subtract* procedure, which involves working right to left, comparable to the *multiply-detach* procedure for the subtraction-first problems. Thus, although an analogue of detachment errors could not be discerned on addition-first equations on the basis of participants' solutions (because the steps yielded a correct solution), data from participants' written procedures suggest that the spacing manipulation did in fact influence the correct procedures that participants used for the addition-first equations.

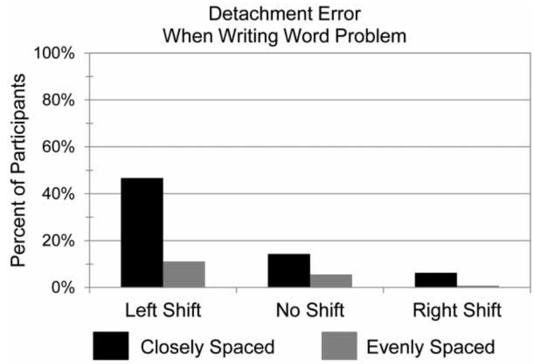

Figure 2. *Percentage of participants who generated story problems that manifested detachment errors, by condition.*

Did the spatial manipulations influence the participant-generated story problems?

We also wished to know whether the spatial manipulations influenced participants' interpretations of the conceptual structure of the problems. To do so, we examined detachment and left-to-right errors in the story problems that participants generated in response to the equation prompt.

Detachment errors. Figure 2 presents the proportion of participants in each condition who generated story problems that reflected detachment errors. To analyse these data, we utilized logistic regression with first operator position and operand spacing as fixed effects.

The best fitting model included main effects of both operand spacing and first operator position. A greater proportion of participants generated story problems with detachment errors in the closely spaced conditions than in the evenly spaced conditions, yielding a main effect of operand spacing, $z = 2.31$, $p = .02$. A greater proportion of participants also generated story problems with detachment errors in the left-shift condition than in the right-shift condition, $z = 2.22$, $p = .03$. Indeed, a model that included first operator position provided a better fit to the data than the nested model that did not include first operator position, change in deviance = 8.26, $p = .02$. The odds that participants produced a detachment error were 5.45 times greater in the closely spaced condition than in the evenly spaced condition, 95% CI [1.42, 27.36]. Likewise, a model that included operand spacing provided a better fit to the data than the nested model that did not include operand spacing, change in deviance = 6.27, $p = .01$. The odds that participants produced a detachment error were 12.07 times greater in the left-shift condition than in the right-shift condition, 95% CI [1.91, 238.74]. A model that included the interaction between minus sign position and operand spacing did not provide a better fit than the model without the interaction, change in deviance = 0.51, $p = .77$.

We also examined whether those participants who generated story problems with detachment errors had been more likely to produce such errors when solving equations. Indeed, the 13 participants who generated story problems that would lead to detachment errors made significantly more detachment errors in equation solving, on average, than the 72 participants who generated story problems that did not contain detachment errors (M detachment errors on equation solving = 6.3, $SE = 1.2$, $Mdn = 8$ vs. $M = 0.46$, $SE = 0.17$, $Mdn = 0$), Mann–Whitney $U = 803$, $p < .001$, $r = .57$. Note that six participants did not respond to the item asking them to generate a story problem; these participants were excluded from this analysis.

Left-to-right errors. We also examined whether participants generated story problems that reflected left-to-right errors. Five participants did so, and all five were in the evenly spaced condition, suggesting again that the closely spaced condition protected participants from making such errors. Of these five participants, three were in the left-shift condition, and two were in the no-shift condition.

Participants who produced story problems that manifested left-to-right errors were also more likely to have produced such errors when solving equations. The five participants who generated story problems with left-to-right errors were much more likely to have made left-to-right errors in equation solving than were participants who generated story problems that did not manifest left-to-right errors (on equation solving, $M = 12.6$, $SE = 3.0$, $Mdn = 14$ vs. $M = 0.43$, $SE = 0.28$, $Mdn = 0$).

Did the spatial manipulations influence how participants solved a story problem?

Finally, we examined whether the spatial manipulations affected participants' performance on solving a story problem. Participants could have perseverated in making errors like those they had made in the equation-solving portion of the study. However, we expected that, instead, the story context would protect participants from making such errors. Indeed, performance on the story-problem-solving item was uniformly high across conditions, with 90% of the participants solving the problem correctly. Only four participants made a detachment error in solving the story problem.

These findings suggest that the story context "shielded" participants from making the detachment errors that some participants frequently produced in the equation-solving portion of the study. To test this idea statistically, we used a McNemar change test. Indeed, participants were significantly more likely to make a detachment error on the equations but not the story problem, than to do the reverse—that is, to make a detachment error on the story problem but not the equations, $\chi^2(1) = 4.57$, $p = .03$.

DISCUSSION

In this study, we manipulated the spacing of the operands and the position of the first operator in multioperation equations involving subtraction. Both of the spatial manipulations affected participants' equation-solving procedures, as revealed in their problem solutions and written work. In addition, both spatial manipulations also influenced participants' interpretations of the conceptual structure of the equations, as revealed in the story problems they generated.

Spatial factors affected participants' equation-solving procedures

We identified participants' equation-solving procedures on the basis of the solutions they reached and the written work they showed. The subtraction-first equations afforded detachment errors, in which participants incorrectly solved $a-(b+c\times d)=\_\_$, rather than the given equations. Both addition-first and subtraction-first problems also afforded left-to-right errors, in which participants simply performed the operations in left to right order. We also considered the two procedures that participants used to arrive at correct solutions (*multiply-subtract* and *multiply-add negative)*. We discuss the effects of the spatial manipulations on each of these outcomes, in turn.

Detachment errors. We hypothesized that closely spacing the final three terms and left-shifting the minus sign would encourage participants to group the last three terms, and this should lead to more detachment errors in subtraction-first problems. Indeed, this is just what we found. Thus, participants' solution procedures depended not only on the meanings of the symbols, but also on perceptual features of the equations, which were affected by the spatial manipulations.

Left-to-right errors. The spatial manipulations also influenced solvers' tendencies to ignore the rules for order of operations altogether and simply solve from left to right. Close spacing of the final three terms protected participants from making this type of error. The close spacing appeared to focus participants' attention on the perceptual group that contained the multiplication operation, emphasizing its precedence and thereby preventing left-to-right errors. Thus, close spacing of the final three terms promoted detachment errors, but prevented left-to-right errors.

Correct solution procedures. In the closely spaced conditions, many participants followed the traditional rules for order of operations, solving the multiplication operation first and then the subtraction operation (i.e. $25-3$ in the equation $25-3+2\times5=\_\_$) before performing the addition operation (i.e. $3+10$). This tendency was especially strong in the closely spaced, right-shift condition—64% of participants in this condition showed written work that reflected this multiply-subtract procedure at least once (see Table 2). These participants seemed to construe the right-shifted minus sign as "part" of the closely spaced perceptual group, and therefore, after solving the multiplication, they proceeded left to right *within* that group, solving the subtraction operation and then the addition operation.

A smaller number of participants in the right-shift conditions (both evenly and closely spaced) seemed to "bind" the right-shifted minus sign to the subsequent operand—that is, treating the problem as $25+[(-3)+2\times5]$. As seen in Table 2, 17% of participants in the evenly spaced and 14% in the closely spaced, right-shift conditions showed written work that reflected this multiply-add negative procedure (as did as a handful of participants in other conditions). For these participants, the minus sign seemed to activate the notion of adding a negative number; they added −3 to 10 after performing the multiplication.

Thus, the spatial manipulations affected the procedures that participants used to solve the equations. The analyses of participants' written work highlight *variability* in the procedures that undergraduate students use to solve multioperation arithmetic problems—a phenomenon that has been observed in a range of other mathematical domains (see Alibali & Sidney, in press, for a review). The current findings point to spatial features of the equations as one factor that influences this variability.

Spatial factors affected participants' conceptual interpretations of arithmetic equations

As expected, both types of spatial manipulations also influenced the likelihood that participants generated story problems that manifested detachment errors. Thus, the manipulations affected participants' conceptual interpretations of the equations and not only their equation-solving procedures.

Those participants who generated story problems manifesting detachment errors also had made more detachment errors when solving equations. It is possible that repeatedly applying a particular procedure when solving the equations led participants to generate a conceptual structure that they then used to interpret the equation on the story-generation item. However, in some cases, solvers generated story problems that incorporated detachment errors, even when they had solved many of the equations correctly. Thus, spatial manipulations can affect conceptual interpretations, even when an incorrect procedure has not been "primed" by repeated practice. Instead, it appears that solvers' interpretations are sensitive to the same factors as those that influence the procedures they use in solving equations.

In contrast to the story problem generation task, participants rarely made detachment errors when *solving* story problems. The story context seemed to help participants generate an alternative, mathematically correct conceptual structure and consequently shielded them from making detachment errors.

Theoretical implications

This study has important implications for understanding the cognitive processing of numbers and operations. First, our analyses revealed that participants used two distinct approaches to correctly solve subtraction-first equations (multiply-subtract and multiply-add negative), and they raise new questions about the sources and consequences of variability in solution procedures, both within and across participants. Second, this study is the first to show that spatial factors can affect participants' conceptual interpretations of multioperation equations. The story generation task required participants to go beyond simply applying a procedure—instead, it required them to act on their conceptual interpretation of the meaning of the equation.

Our findings also contribute to knowledge of cognitive processes in subtraction. This work complements the substantial body of research suggesting that subtraction can be cognitively challenging, even for adults. We found that, like middle school students (Herscovics & Linchevski, 1994), adults sometimes detach the minus sign from the following operand when solving multioperation equations. We also showed that spatial manipulations could either promote or prevent this error, depending on whether they incorporated the subtraction operation into the relevant perceptual group or not.

More generally, our findings highlight the need for a more process-based understanding of how conceptual and perceptual information are integrated in mathematical thinking. Participants varied in how frequently they produced detachment errors and left-to-right errors, with some participants making none and others making many. Thus, it appears that irrelevant spatial features influenced some participants more than others. Moreover, when they solved the equations correctly, some participants activated knowledge of subtraction as addition of a negative number. At present, however, little is known about what factors might influence which sorts of information solvers focus on, activate, or privilege in problem solving. Contextual factors, individual differences in cognitive abilities, and educational history may all be relevant.

Along these lines, one valuable next step would be to analyse participants' equation-solving steps in greater detail. Here, we examined equation-solving processes in participants who showed their work; however, despite our instructions, many participants did not consistently do so. If we had used talk aloud protocols or methods such as eye tracking, we might have obtained additional process-level information that could support or constrain our interpretations (e.g. explicit evidence regarding which portions of the equations participants focused on and in what order). Such evidence could further bolster the claim that perceptually grouped operations are likely to be performed first (Landy & Goldstone, 2010), and it would align with eye-tracking data showing that people tend to focus on narrow spaces early in a trial and move toward wider spaces toward the end of the trial (Landy et al., 2008).

Eye tracking research has also shown that people extract syntactic structure from mathematical equations quite rapidly (Schneider, Maruyama, Dehaene, & Sigman, 2012), but even so, spatial manipulations can still influence how equations are processed, as the present findings show.

It would also be worthwhile to study the effects of spatial manipulations on younger students, including those who are making the transition from arithmetic to algebra. At present, little is understood about how students utilize perceptual information as they progress in their mathematics education and as they begin to use the minus sign in diverse ways (i.e. to denote negative numbers as well as subtraction). The results from the story problem generation task suggest that spatial features of equations do not lead simply to careless errors, but instead lead to specific misconceptions about the relationships among the mathematical symbols.

Implications for education

This work is part of a growing body of research that highlights the potential value of supporting learners' appropriate perceptual encoding in educational contexts (e.g. Goldstone et al., 2010; Prather & Alibali, 2011). For example, instruction that provides students with opportunities for perceptual learning leads to substantial gains in problem solving across a range of mathematical domains, including measurement, fractions, and algebra (Kellman et al., 2008, 2010).

In light of this past research, we suggest that instructional materials should capitalize on the effects of perceptual grouping on mathematical processing. For example, spatial manipulations could be used to support students' learning of the rules for order of operations. Appropriate perceptual groups could help prevent precedence errors and detachment errors, and perceptual support could be faded as students gain proficiency.

Spatial cues could also be utilized in teaching about negative integers. Our findings indicate that some students' concepts of negative numbers were activated by the dual nature of the minus sign, especially in the right-shift condition, where the minus sign was closer to the subsequent operand. Thus, spatial features, such as the proximity of the minus sign to the subsequent numeral, may influence whether learners activate two concepts of the minus sign. This phenomenon could be leveraged in instruction in various ways, depending on the instructional goals. For example, in a lesson about operations with negative numbers that stresses the arithmetic equivalence of subtracting a positive number and adding a negative number, using a right-shifted minus sign might help students activate the notion of negative numbers as well as the notion of subtraction.

Conclusion

The present study has shown that spatial cues that afford perceptual groups affect participants' solutions to arithmetic equations and also affect their interpretations of the conceptual structure of those equations. These findings are important, both theoretically and practically, because mathematics uses a symbol system that incorporates spatial features in its notation. More broadly, our findings highlight the importance of perceptual factors in mathematical reasoning, and they call for a deeper consideration of the spatial characteristics of the symbolic expressions used in mathematics.

Original manuscript received 31 July 2013
Accepted revision received 12 December 2013
First published online 15 April 2014

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APPENDIX A

Equations used in the study, in the order that they were used

Note that one addition-first equation was inadvertently presented twice [6 participants (6.5%) answered the two items differently].

$28 - 7 + 2 \times 3 = \_\_\_\_$
$30 - 9 + 5 \times 2 = \_\_\_\_$
$14 + 15 - 7 \times 3 = \_\_\_\_$
$14 + 24 - 3 \times 6 = \_\_\_\_$
$23 - 5 + 2 \times 5 = \_\_\_\_$
$24 - 7 + 6 \times 2 = \_\_\_\_$
$13 + 11 - 3 \times 6 = \_\_\_\_$
$15 + 14 - 9 \times 3 = \_\_\_\_$
$26 - 8 + 3 \times 5 = \_\_\_\_$
$25 - 11 + 2 \times 6 = \_\_\_\_$
$12 + 14 - 5 \times 3 = \_\_\_\_$
$13 + 11 - 3 \times 6 = \_\_\_\_$
$29 - 12 + 3 \times 5 = \_\_\_\_$
$29 - 19 + 4 \times 2 = \_\_\_\_$
$13 + 11 - 7 \times 2 = \_\_\_\_$
$13 + 15 - 6 \times 3 = \_\_\_\_$
$30 - 14 + 4 \times 3 = \_\_\_\_$
$19 - 5 + 3 \times 2 = \_\_\_\_$

APPENDIX B

Distribution of the total number of detachment errors, by condition

Shown in the figure are the number of participants making *x* detachment errors for each combination of operand spacing and first operator position.