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# Relations between patterning skill and differing aspects of early mathematics knowledge<sup>☆</sup>



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## ABSTRACT

Patterns are often considered central to early mathematics learning; yet, the empirical evidence linking early pattern knowledge to mathematics performance is sparse. In the current study, 36 children ranging in age from 5 to 13 years old ( $M = 9.1$  years) completed a pattern extension task with three pattern types that varied in difficulty. They also completed three math tasks that tapped calculation skill and knowledge of concepts. Children were successful on the pattern extension task, though older children fared better than younger children, potentially due in part to their explanations that considered both dimensions of the pattern (shape and size). Importantly, success on the pattern extension task was related to mathematics performance. After controlling for age and verbal working memory, patterning skill predicted calculation skill; however, patterning skill was not associated with knowledge of concepts. Results suggest that patterning may play a key role in the development of some aspects of early mathematics knowledge.

## 1. Introduction

Patterns are ubiquitous in early learning environments, particularly in mathematics. Whether creating an alternating sequence of blue and red cubes or determining the next number in a function table, children often must make predictions based on the rule governing a pattern. Some have argued that “patterns are what mathematics is all about” and have defined mathematics as the science of patterns (Steen, 1988, p. 616). Further, recent empirical research has provided both correlational and causal evidence relating patterning skills to mathematics achievement (Kidd et al., 2013; Kidd et al., 2014; Papic, Mulligan, & Mitchelmore, 2011; Rittle-Johnson, Fyfe, Hofer, & Farran, 2016). However, little is known about how patterning skill relates to different types of mathematics knowledge – for example, calculation skill versus knowledge of specific math concepts. The goals of the current study

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were (1) to document children's performance on a novel pattern extension task, and (2) to examine relations between patterning skill and performance on three mathematics tasks that tap both calculation skill and knowledge of concepts.

One of the first types of patterns that children engage with are repeating patterns – linear patterns with a unit that repeats (e.g., ABABAB; National Association for the Education of Young Children, 2014; National Council of Teachers of Mathematics, 2006). The unit usually contains two or three elements (e.g., AB, ABB, ABC) that vary on a key dimension, such as color (e.g., red-blue-red-blue-red-blue), shape (e.g., square-circle-square-circle-square-circle), or size (e.g., big-small-big-small-big-small). Empirical research has provided correlational and causal evidence relating repeating patterning knowledge to mathematics achievement. For example, in a longitudinal sample, repeating pattern skill at the end of pre-k, kindergarten, and first grade was a unique predictor of mathematics achievement in fifth grade, even after controlling for other mathematics, reading, cognitive, and demographic measures (Rittle-Johnson et al., 2016; see also Fyfe & Rittle-Johnson, 2016).

Several intervention studies have documented relations between repeating pattern knowledge and mathematics performance. In a non-experimental, classroom-based study involving instruction on repeating patterns, Warren and Cooper (2007) examined children's discussions of target patterning and ratio tasks. They argued that repeating patterns served as an "effective bridge" (p. 14) to understanding ratios (e.g., if the ratio of juice to water in jug 1 is 2:4 and the ratio of juice to water in jug 2 is 4:8, which jug has the stronger juice?). In a quasi-experimental study, Papic et al. (2011) examined the effectiveness of a six-month repeating pattern intervention. Children in a preschool that used the intervention exhibited better skills with growing patterns at the end of the year than children in a preschool that did not. Further, children who received the intervention scored higher on a standardized number assessment at the end of kindergarten than children who did not.

Two randomized controlled trials provide causal evidence that instruction on repeating patterns supports mathematics knowledge. In two separate studies, Kidd and colleagues (Kidd et al., 2013; Kidd et al., 2014) randomly assigned struggling first-grade students to receive supplemental instruction in patterning, mathematics, reading, or social studies. The patterning instruction included repeating patterns and other pattern types (e.g., symmetrical patterns, rotation patterns) and occurred three times a week over a six-month period. The patterning intervention resulted in comparable or better performance on standardized mathematics assessments, relative to the other interventions.

Clearly, an emerging body of research suggests that children's repeating pattern knowledge may play a pivotal role in their mathematics performance. However, the number of studies documenting this relation remains small. More importantly, the few studies that have documented this pattern-math relation have relied primarily on global measures of mathematics achievement. The one exception by Warren and Cooper (2007) relied on qualitative descriptions of classroom episodes. It is possible that patterning skill may be differentially related to different aspects of math knowledge. For example, a distinction is often made between children's ability to calculate an answer and the conceptual knowledge that underpins correct strategies (e.g., Crooks & Alibali, 2014; Rittle-Johnson & Alibali, 1999; Siegler, 2000). To date, no studies have investigated whether children's pattern skill is associated with different aspects of mathematics knowledge in similar or different ways.

The primary goal of the current study was to examine the relations between children's patterning skill and different aspects of their mathematics performance. We selected three math tasks that allowed us to examine children's calculation skill and their knowledge of important mathematical concepts. First, we included standard arithmetic problems in an operations-equals-answer format (e.g.,  $2 + 4 + 5 + 2 = \_$ ). These problems are routine for young children, and they can be used to tap calculation skill. Second, we included mathematical equivalence problems, which contain operations on both sides of the equal sign (e.g.,  $3 + 4 + 6 = 3 + \_$ ). To solve these problems correctly, children must realize that the equal sign indicates an equivalence relation – that is, the amounts on the two sides must be the same. Thus, these problems tap children's understanding of mathematical equivalence, a key concept that children acquire during their elementary school years, but that is challenging for many children in the US and Canada (e.g., Falkner, Levi, & Carpenter, 1999; Kieran, 1981; McNeil & Alibali, 2005). Third, we included inversion problems, in which the same value is added and subtracted (e.g.,  $4 + 7 - 7 = \_$ ). Children can solve these problems correctly either by performing each operation (e.g., "4 plus 7 is 11, and 11 minus 7 is 4") or by recognizing the inversion principle, which holds that adding and subtracting the same quantity results in no net change (e.g., "the 7s cancel so it keeps the 4 the same"). Thus, these problems tap children's understanding of mathematical inversion, another key concept that children typically acquire during their elementary school years, but that is challenging for many children (Gilmore & Bryant, 2008; Robinson, Ninowski, & Gray, 2006). Both equivalence and inversion problems have been widely used in the literature on the development of conceptual knowledge (see Crooks & Alibali, 2014, for a review).

We examined children's performance on these three math tasks as it related to their repeating pattern knowledge. Research has documented children's performance on a variety of repeating patterning tasks – including copying a model pattern with the same or different materials and identifying the unit that repeats (e.g., Clements, Sarama, & Liu, 2008; Fyfe, McNeil, & Rittle-Johnson, 2015; Mulligan & Mitchelmore, 2009; Rittle-Johnson, Fyfe, McLean, & McEldoon, 2013; Rittle-Johnson, Fyfe, Loehr, & Miller, 2015; Warren & Cooper, 2006). Here, we focused on a pattern extension task, which involves continuing a model pattern (e.g., "what comes next?"). Pattern extension is a common and popular patterning task for young children (Economopoulos, 1998; Rittle-Johnson et al., 2015). Further, it is accessible at an early age, but with substantial variability and room to improve (e.g., Aubrey, 1993; Starkey et al., 2004; Rittle-Johnson et al., 2013). Thus, it was well suited to investigate pattern knowledge in early childhood. In previous work, item difficulty has often been defined by the pattern task (e.g., copying versus extending). We opted instead to use a single pattern task (i.e., pattern extension) and to vary the difficulty of the type of patterns used within the task. Thus, a secondary goal of the current study was to document children's performance on this novel pattern extension task. To do so, we examine both children's overall level of success and the nature of the strategies they used in extending different types of patterns.

In the current study, children saw patterns with elements that varied in shape (circle or square) and size (big or small), and had to

Same-Same Patterns	Same-Different Patterns	Different-Different Patterns
Shape: ABBABBA Size: ABBABBA	Shape: ABAABAA Size: ABBABBA	Shape: AABAABA Size: ABABABA
● ■ ■ ● ■ ■ ● _	● ■ ● ● ■ ● ● _	■ ■ ● ■ ■ ● ■ _

Fig. 1. Sample pattern extension items for each pattern type.

indicate which of four elements would come next in each sequence. For example, on one item children saw the following pattern at the top of the page: ● ■ ■ ● ■ ■ ●. They were then presented with the four possible choices (i.e., big square, small square, big circle, small circle) and asked to point to the one that would come next and to explain their selection. We included three different pattern types that varied in complexity (see Fig. 1). Specifically, we varied whether the unit of repeat was the same for shape and size, both in terms of the unit length (two or three items) and the specific type of pattern (AAB, ABA, ABB, or AB). We asked children to provide a verbal explanation for each selection so that we could examine their responses, both qualitatively and quantitatively, as a function of pattern type and age. Prior research on patterning suggests that children's explanations may reveal distinct approaches to the patterning task that are associated with variations in task performance (e.g., Fyfe et al., 2015; Rittle-Johnson et al., 2015).

In addition to assessing children's patterning and mathematics skills, we also measured their working memory capacity. Past research has suggested that working memory – a short-term system that enables individuals to actively select, regulate, and process a limited amount of task-relevant information (Baddeley & Logie, 1999) – plays a role, both in mathematics performance (e.g., Peng, Namkung, Barnes, & Sun, 2016; Raghobar, Barnes, & Hecht, 2010) and in patterning (Miller, Rittle-Johnson, Loehr, & Fyfe, 2016; Rittle-Johnson et al., 2013). In light of this past work, we considered whether relations between patterning skill and mathematics might be due to their shared relations with working memory.

In brief, our goals were: (1) to document children's performance on a novel pattern extension task, (2) to examine relations between children's patterning skill, calculation skill, and knowledge of equivalence and inversion concepts, and (3) to examine whether potential relations between patterning and math are due to their shared reliance on working memory.

## 2. Method

### 2.1. Participants

Participants were 36 children (47% female) ranging in age from 5 to 13 years old ( $M = 9.1$  years,  $SD = 2.3$ ,  $range = 5.3$ – $13.4$ ). Children were primarily from white, middle- to upper-class households (78% White, 17% Black, 5% Hispanic). Based on parent report, participants' mothers had completed an average of 15.6 years of schooling ( $SD = 2.9$ ,  $range = 12.0$ – $21.0$ ), which is approximately equivalent to a four-year college degree.

### 2.2. Materials

There were five key tasks: a working memory task, a pattern extension task, and three mathematics tasks (standard arithmetic problems, inversion problems, and equivalence problems). See the Appendix A for a list of the items in the order they were presented. All the pattern and mathematics items were printed on  $11 \times 17$  sheets of white, laminated paper.

#### 2.2.1. Working memory

All children completed the Competing Language Processing Task (CLPT, Gaulin & Campbell, 1994), a measure of verbal working memory. This measure has been validated for children between 6 and 12 years of age. In this task, children listen to sets of one to six sentences and are asked to verify the truth of each sentence (in a yes/no judgement), while retaining the last word in the sentence for later recall. After verifying the sentences in a set, children are asked to list the last word of each sentence in the set. The percent of words recalled correctly is the score.

#### 2.2.2. Pattern task

The pattern task was used to evaluate children's ability to predict what comes next in a pattern. On the top of each page, seven elements formed the pattern sequence and were followed by a blank to indicate the continuation of the pattern. The elements in each given pattern were solid black and varied in shape (circle or square) and size (big or small). On the bottom of each page, the four possible alternatives for what could come next (i.e., big square, small square, big circle, small circle) were presented. Children were instructed to point to the element that comes next in the pattern, and to explain how they arrived at their selection ("How did you get that?").

There were three practice items followed by 24 target items. On each target item, two patterns were embedded in the given pattern sequence, one for shape and one for size. These patterns varied in unit length (two or three elements) and in unit type (AAB, ABA, ABB, or AB). These variations were implemented in three pattern types: (1) Same-Same patterns, in which unit length and unit type were the same for shape and size, (2) Same-Different patterns, in which unit length was the same for shape and size, but unit type differed, and (3) Different-Different patterns, in which unit length and unit type differed for both shape and size. Examples are

**Table 1**  
Performance on the pattern extension task by pattern type.

Pattern Type	% Correct (SD)	% at Mastery
Same-Same	91 (20)	72
Same-Different	91 (19)	72
Different-Different	66 (28) <sup>***</sup>	19 <sup>***</sup>

<sup>\*\*\*</sup>  $p < 0.001$ . Performance on different-different items was significantly lower than performance on same-same and same-different items. Percent correct was analyzed using paired samples *t*-tests. Percent of children at mastery was analyzed using McNemar tests.

presented in Fig. 1. There were eight items of each pattern type presented in a mixed order.

### 2.2.3. Math tasks

There were fifteen math problems in total (see Appendix). On each problem, children were instructed to solve the problem and write the solution in the blank. They were then asked to explain how they arrived at that solution (“How did you get that?”). Three of the problems were *standard arithmetic problems*, which presented four addends in an operations-equals-answer format (e.g.,  $2 + 4 + 5 + 2 = \_$ ). Six were *equivalence problems*, which are equations with operations on both sides of the equal sign (e.g.,  $3 + 4 + 6 = 3 + \_$ ). Six were *inversion problems*, in which the same number was added to and subtracted from a target number (e.g.,  $4 + 7 - 7 = \_$ ).

## 2.3. Coding

### 2.3.1. Pattern task

On the pattern task, children’s responses were coded as correct if they selected the element that would come next in the given pattern. Children’s verbal explanations were coded into one of eight categories using a system adapted from one used in previous research (Rittle-Johnson et al., 2013). The system was based on detailed transcriptions of children’s speech and gesture when describing patterns. Explanation types that were frequent and consistent across children were grouped into categories. In the current study, we categorized children’s explanations based on these existing common codes (e.g., labeling the items of a pattern in order), but we also attended to whether children described one or both varying dimensions (shape and size). The codes, their descriptions, and examples are presented in Table 2. To establish inter-rater reliability, a second rater coded 30% of explanations; agreement was very high (0.95).

### 2.3.2. Math tasks

On the arithmetic and inversion problems, children’s solutions were coded as correct only if they were exactly correct. For inversion problems, we also coded children’s strategies based on their explanations using a system from previous research (e.g., Siegler & Stern, 1998). We were primarily interested in whether children reported the conceptual *inversion shortcut* – that is, whether they realized that the answer would be the first number, without actually adding and subtracting the second and third numbers (e.g., “the sevens cancel so it keeps the four the same”). Children also reported calculation strategies, in which they added the first two numbers and subtracted the third (e.g., “four plus seven is eleven and eleven take away seven is four”), as well as ambiguous strategies (e.g., “four take away seven would make it to go ten”). Inter-rater agreement on whether the conceptual shortcut was used was high (0.97).

For equivalence problems, we inferred whether children used a correct strategy based on whether they provided a correct problem solution, that is, a solution that made both sides of the equation equal. We further coded the specific type of correct or incorrect strategy into one of seven categories based on their verbal explanations, using a system developed in previous research (Perry,

**Table 2**  
Frequency of explanation use on the pattern task by pattern type.

Explanation Type	Description and Example	SS	SD	DD	Total
<i>Labels Elements in Order: Both Dimensions</i>	Labels elements in order by shape and size (“big square, little circle, little circle”)	17	32 <sup>a</sup>	19	23
<i>Labels Elements in Order: One Dimension</i>	Labels elements in order by shape or size (“square, circle, circle”)	23 <sup>b</sup>	10	11	15
<i>Match</i>	Matches a previous instance (“the big square here matches the big square here, and after it was a little circle”)	19	22	19	20
<i>Names Characteristic: Both Dimensions</i>	Names sizes and shapes of elements (“big squares and little circles”)	1	2	5	3
<i>Names Characteristic: One Dimension</i>	Names sizes or shapes of elements (“squares and circles”)	5	1 <sup>a</sup>	6	4
<i>Pattern Word</i>	Mentions following a pattern with nothing more specific (“I just followed the pattern”)	7	7	7	7
<i>Vague</i>	Attempts an explanation, but is unclear (“by looking behind”)	11	9	14	11
<i>No Verbal Response</i>	No verbal answer or gives statement of uncertainty (“I don’t know”)	17	16	19	17

Note: SS = same-same items, SD = same-different items, DD = different-different items.

<sup>a</sup> Percent use on SD items was significantly different than on SS and DD items,  $ps < 0.05$ .

<sup>b</sup> Percent use on SS items was significantly different than on SD and DD items,  $ps < 0.05$ .

**Table 3**  
Frequency and success rate of explanation type on the pattern task by age.

Strategy	% Use Across Items		% Successful Items	
	Younger (age 5–8)	Older (age 9–13)	Younger (age 5–8)	Older (age 9–13)
<i>Labels Elements in Order: Both Dimensions</i>	13 <sup>a</sup>	34	85 <sup>b</sup>	98
<i>Labels Elements in Order: One Dimension</i>	14	15	91	94
<i>Match</i>	13	28	89	93
<i>Names Characteristic: Both Dimensions</i>	3	3	50	75
<i>Names Characteristic: One Dimension</i>	5	2	78	50
<i>Pattern Word</i>	13	1	78	100
<i>Vague</i>	16	6	75	62
<i>No Verbal Response</i>	23	11	53	63

<sup>a</sup> Percent use was significantly lower for younger than older children,  $p < 0.05$ .

<sup>b</sup> Percent successful items was marginally lower for younger than older children,  $p < 0.10$ .

Church, & Goldin-Meadow, 1988). Codes and sample explanations are presented in Table 3. Inter-rater agreement on whether solutions were correct or not was perfect (1.00), and agreement on specific strategies was high as well (0.93).

Given our interest in the relation between patterning and mathematics, we created two summary math scores. First, we created a *Calculation Score* by summing the number of problems solved correctly on the three standard arithmetic problems and on the six inversion problems. Correctness on these problems is determined by the accuracy of the solution and can be obtained with calculation skill. Second, we created a *Concepts Score* by summing the number of inversion problems solved with the shortcut strategy and the number of equivalence problems solved correctly (which includes the equalize, add-subtract, and grouping strategies in Table 4).

#### 2.4. Procedure

Children participated individually in two sessions conducted by a female experimenter in a quiet laboratory space. The first session was part of a previous study and took place approximately six months prior to the second session. In the first session, children completed the verbal working memory task, along with several other cognitive measures not relevant to the current research questions. In the second session, children were videotaped as they completed the pattern and mathematics tasks. They first completed the pattern task, then the arithmetic and equivalence problems in mixed order (see Appendix), and then the inversion problems.

### 3. Results

#### 3.1. Performance on the pattern extension task

Children were quite successful at selecting the correct element to extend each pattern. Across all 24 items, average percent correct was 83% ( $SD = 19\%$ ), which is significantly above chance (i.e., 25%),  $t(35) = 18.57, p < 0.001$ . All but three children (33 out of 36, 92%) selected the correct element on more than half of all items, and 17% of children selected the correct element for all 24 items.

However, performance varied by pattern type. As shown in Table 1, performance was nearly identical for same-same and same-different items, but lower on the different-different items. Indeed, percent correct on same-same items was not statistically different from percent correct on same-different items,  $t(35) = 0.27, p = 0.79$ , but both were significantly higher than percent correct on different-different items,  $ts(35) > 24.00, ps < 0.001$ . Similarly, percent of children at mastery (i.e., who selected the correct element on all 8 items of a given type) was identical on the same-same and same-different items, but significantly lower on the different-different items,  $ps < 0.001$ . This pattern of results was highly consistent across children. For example, among children who missed at

**Table 4**  
Frequency of strategy explanations on the mathematical equivalence problems.

Strategy	Sample verbal explanation (3 + 4 + 6 = 3 + _)	% use across items		
		Younger (age 5–8)	Older (age 9–13)	All
Correct				
<i>Equalize</i>	3, 4, and 6 is 13, and 3 plus 10 is 13.	2	28	14
<i>Add-Subtract</i>	3 plus 4 is 7, and 7 plus 6 is 13. 13 minus 3 is 10.	0	6	3
<i>Grouping</i>	The 3s are the same, and then 4 plus 6 is 10.	4	14	8
Incorrect				
<i>Add All</i>	I added the 3, the 4, the 6, and the 3 together.	49	29	40
<i>Add-to-Equal</i>	Well I know 3 plus 4 is 7 and 7 plus 6 is 13.	24	11	18
<i>Carry</i>	It's a copy. You see the 4 here, so you put 4.	3	12	7
<i>Other</i>	All these take away to zero.	19	1	11

least one item, 87% scored lowest on the different-different items, and 53% had identical scores on the same-same and same-different items.

Performance on the pattern extension task also varied with age and working memory. Age in years was positively correlated with percent correct on all 24 pattern items,  $r(34) = 0.51$ ,  $p = 0.001$ . Further, when we split children into a younger group (ages 5–8,  $n = 19$ ) and an older group (ages 9–13,  $n = 17$ ), the older group outperformed the younger group ( $M_s = 91\%$  vs.  $75\%$ ),  $t(34) = 2.70$ ,  $p = 0.01$ . This difference held for each of the three pattern types, though the advantage for older children was somewhat greater on the more difficult items (10% difference on same-same items, 16% difference on same-different items, and 21% difference on different-different items). Despite these age differences, the younger children were still quite successful, solving 87% of the same-same items, 83% of the same-different items, and 57% of the different-different items correctly, each of which is significantly above chance (i.e., 25%),  $p_s < 0.01$ .

Verbal working memory was also positively correlated with percent correct across all 24 pattern items,  $r(34) = 0.47$ ,  $p = 0.004$ . This relation held after controlling for age,  $\beta = 0.32$ ,  $SE = 0.15$ ,  $p = 0.04$ , though age remained a significant predictor as well,  $\beta = 0.39$ ,  $SE = 0.15$ ,  $p = 0.02$ .

### 3.2. Explanations on the pattern extension task

On each pattern extension item, children were asked to explain how they arrived at their selection (“How did you get that?”). As shown in Table 2, children’s descriptions fell into one of eight categories. As in prior work (Rittle-Johnson et al., 2013), the most common explanation (~40% of all items) was to label the elements in order, either by naming their shape and their size (i.e., *labels-both-dimensions*; “big square, little circle, little circle, big square, little circle, little circle) or just their shape or size (i.e., *labels-one-dimension*). The next most common explanation (20% of all items) was to relate the final element to a matching element in the pattern (i.e., *match*; “this big square at the end matches this big square at the beginning, and after the first big square there was a little circle”). The remaining five categories represented non-pattern-based explanations (i.e., names both characteristics of the elements without reference to position, names one characteristic of the elements without reference to position, states they followed a pattern without specifying the pattern, gives vague explanations, or refuses to answer).

Overall, children were variable in their explanations, providing an average of 3.9 different explanation types across the 24 items ( $SD = 1.8$ ,  $min = 1$ ,  $max = 8$ ). However, they were more consistent within each pattern type. For example, for each of the three pattern types, ~70% of children gave the same type of explanation on 6 or more of the 8 items.

There were some differences in explanations across the three pattern types. For example, children labeled elements in order using one dimension (i.e., *labels-one-dimension*) more frequently on same-same items than on other items. Conversely, children labeled elements in order using both dimensions (i.e., *labels-both-dimensions*) more frequently on same-different items than on others. Interestingly, the frequency of using the five non-pattern-based explanations was higher on different-different items ( $M = 51\%$ ,  $SE = 7\%$ ) relative to same-different items ( $M = 36\%$ ,  $SE = 7\%$ ),  $t(35) = 3.21$ ,  $p = 0.003$ , and same-same items ( $M = 41\%$ ,  $SE = 7\%$ ),  $t(35) = 1.94$ ,  $p = 0.06$ , perhaps explaining children’s lower performance on these items.

Two of the eight explanation types were related to success in selecting the correct element to extend the pattern. The frequency of a no-verbal-response explanation was negatively related to percent correct,  $r(34) = -0.55$ ,  $p < 0.001$ , and the frequency of a labels-both-dimensions explanation was positively correlated with percent correct,  $r(34) = 0.33$ ,  $p = 0.05$ . This pattern of results also held when we considered whether children ever provided a given explanation type. Specifically, children who ever provided a no-verbal-response explanation scored lower ( $M = 74\%$ ,  $SE = 6\%$ ) than children who never did ( $M = 88\%$ ,  $SE = 3\%$ ),  $t(35) = 2.32$ ,  $p = 0.03$ , and children who ever provided the labels-both-dimensions explanation scored higher ( $M = 87\%$ ,  $SE = 2.7\%$ ) than children who never did ( $M = 70\%$ ,  $SE = 8.6\%$ ),  $t(35) = 2.46$ ,  $p = 0.02$ .

Age was correlated with the labels-both-dimensions explanation,  $r(34) = 0.36$ ,  $p = 0.03$  (see also Table 3), as was working memory,  $r(34) = 0.34$ ,  $p = 0.04$  (and neither were correlated with any other explanation type). As noted above, age and working memory were also associated with performance, assessed in terms of percent correct; these associations remained significant after taking into account frequency of use of the labels-both-dimensions strategy,  $\beta = 0.45$ ,  $p = 0.01$ , for age, and  $\beta = 0.41$ ,  $p = 0.02$ , for working memory. Thus, age-related and working-memory-related increases in use of the labels-both-dimensions strategy did not fully account for the observed age- and working-memory related improvements in performance.

There were also differences in the probability of success for items on which children provided different explanation types. When children offered one of the three pattern-based explanations (*labels-both-dimensions*, *labels-one-dimension*, and *match*), their average success rate was 92% in each case. That means, for example, if a child provided the labels-both-dimensions explanation on ten items, she likely solved nine of those ten items correctly. In contrast, when children offered one of the five non-pattern-based explanations, they had lower average success rates, which varied from 57% to 84%. Descriptively, there were also age differences in the probability of a success for several of the explanation types (see Table 3). Of interest, older children had a higher rate of success when they offered the labels-both-dimensions explanation, relative to younger children. Thus, not only did older children offer that explanation more frequently than younger children, they were also more successful when they did so.

### 3.3. Performance on the math tasks

#### 3.3.1. Standard arithmetic problems

Children were highly successful on the three standard arithmetic problems (e.g.,  $2 + 4 + 5 + 2 = \underline{\quad}$ ). Average score was 75% ( $SD = 36\%$ ), and over 60% of children solved all three problems correctly, indicating that most children had proficient calculation

skills. Scores on the arithmetic problems were correlated with both age,  $r(34) = 0.64$ ,  $p < 0.001$ , and working memory,  $r(34) = 0.45$ ,  $p = 0.006$ . However, the relation with working memory was no longer significant after controlling for age, whereas age differences were robust. Indeed, older children (ages 9–13) scored nearly perfectly ( $M = 96\%$ ,  $SD = 11\%$ ) and younger children (ages 5–8) scored substantially lower ( $M = 56\%$ ,  $SD = 40\%$ ).

### 3.3.2. Inversion problems

Children were similarly successful at solving the six inversion problems correctly (e.g.,  $4 + 7 - 7 = \_$ ). Average score was 76% ( $SD = 40\%$ ), and over 65% of the children solved all six inversion problems correctly. Recall that we also coded whether children used the conceptual “inversion shortcut” rather than calculating the answer. Children used the shortcut on an average of 31% of problems ( $SD = 43\%$ ), accounting for 42% of all correct responses. The use of the shortcut was bimodal: 60% of children never used it, 23% used it on all six problems, and the remaining 17% of children fell somewhere between these extremes. Neither age or working memory predicted shortcut use. Indeed, the percent of children who ever used the shortcut did not differ statistically in the younger and older groups, 33% vs. 47%,  $\chi^2(1, N = 36) = 0.01$ ,  $p = 0.34$ .

### 3.3.3. Equivalence problems

Performance on the six equivalence problems (e.g.,  $3 + 4 + 6 = 3 + \_$ ) was also bimodal. A full 72% of children solved none of the problems using correct strategies, 22% solved all six with correct strategies, and only two children fell between these extremes. The likelihood of solving at least one problem with a correct strategy was predicted by both age,  $\beta = 0.67$ ,  $p = 0.004$ , and working memory,  $\beta = 0.06$ ,  $p = 0.02$ , though the relation with working memory was no longer significant after controlling for age, whereas age differences were robust. Indeed, the percent of children who ever used a correct strategy was much lower in the younger group than in the older group, 5% vs. 53%,  $\chi^2(1, N = 36) = 10.17$ ,  $p = 0.001$ . As in prior work, the two most common strategies were to add all the numbers together and to add the numbers before the equal sign (see Table 4).

Past research suggests that understanding the inversion principle may be a developmentally earlier achievement than understanding mathematical equivalence in symbolic form (McNeil, 2007; Prather & Alibali, 2009), although to our knowledge, no study to date has examined this. Indeed, many children exhibited knowledge of neither concept (53%) or both concepts (19%), but of those who exhibited knowledge of only one concept, 70% displayed knowledge of inversion but not equivalence, while only 30% displayed knowledge of equivalence but not inversion. Further, the average age of children who understood inversion but not equivalence was lower ( $M = 8.1$ ,  $SD = 0.8$ ,  $n = 7$ ) than the average age of children who understood equivalence, but not inversion ( $M = 9.8$ ,  $SD = 1.2$ ,  $n = 3$ ) or children who understood both equivalence and inversion ( $M = 11.8$ ,  $SD = 1.5$ ,  $n = 7$ ).

## 3.4. Relations between patterning and mathematics

The primary aim of this work was to examine whether children’s patterning skill was related to differing aspects of their math performance, over and above age and working memory. As described in the method, we created two summary math scores, one to reflect children’s calculation skill and another to reflect their knowledge of concepts (equivalence and inversion). As expected based on their performance on the individual tasks, calculation scores were high ( $M = 75\%$ ,  $SD = 36\%$ ) and concept scores were relatively low ( $M = 29\%$ ,  $SD = 37\%$ ). Patterning scores were correlated with both calculation scores,  $r(34) = 0.71$ ,  $p < 0.001$ , and concept scores,  $r(34) = 0.36$ ,  $p = 0.03$ . To better understand these relations, we examined patterning performance, age, and working memory as predictors of math performance, and we considered calculation and concepts separately. In each case, we entered age and working memory into the model first, and then examined whether adding pattern performance improved the fit of the model.

First, we used a step-wise regression to predict calculation scores. In the first step, we entered children’s age and working memory. In the second step, we entered children’s patterning score across all 24 items. This final model was significant,  $F(3, 32) = 18.31$ ,  $p < 0.001$ , and patterning skill significantly predicted calculation scores over and above age and working memory,  $\beta = 0.51$ ,  $SE = 0.13$ ,  $p = 0.001$  (see Table 5). Moreover, including patterning skill resulted in significant improvement in model fit,  $F(1, 32) = 14.89$ ,  $p = 0.001$ , and increased the  $R^2$  value from 0.46 to 0.63. Further, the pattern of results was the same if we used scores on same-same items, same-different items, or different-different items in place of total patterning score.

Second, we performed the same analysis to predict concepts scores. In the first step, we entered children’s age and working memory. In the second step, we entered children’s patterning score across all 24 items. The final model was significant,  $F(3, 32) = 5.07$ ,  $p = 0.006$ . However, in this case, patterning skill did not significantly predict concepts scores over and above age and working memory,  $\beta = 0.07$ ,  $SE = 0.17$ ,  $p = 0.71$  (see Table 5). Including patterning skill did not result in significant improvement in model fit,  $F(1, 32) = 0.14$ ,  $p = 0.71$ , and the  $R^2$  value did not change in any substantive way. The pattern of results was the same if we used scores on same-same items, same-different items, or different-different items in place of a total patterning score.

We also explored whether the absence of association between patterning and knowledge of concepts was the case for each of the two target concepts (inversion vs. equivalence) on its own. To do so, we used two logistic regression models, one to predict the log of the odds of children solving any of the inversion problems via the shortcut strategy and another to predict the log of the odds of children solving any of the equivalence problems using a conceptually correct strategy. In both models, we included age, working memory, and patterning skill. As in the overall analysis, patterning was not a significant predictor of knowledge of the inversion concept,  $\hat{\beta} = 0.44$ ,  $SE = 0.54$ ,  $p = 0.42$ , or of the equivalence concept,  $\hat{\beta} = 1.75$ ,  $SE = 1.32$ ,  $p = 0.19$ . In neither case did including patterning yield a significant improvement in model fit:  $\chi^2(1, N = 36) = 0.73$ ,  $p = 0.39$  for inversion,  $\chi^2(1, N = 36) = 2.37$ ,  $p = 0.12$  for equivalence.

**Table 5**  
Summary of step-wise regression models predicting calculation and concepts scores.

Variable	<i>B</i>	<i>SE B</i>	$\beta$	$\Delta R^2$	Adjusted $R^2$
Predicting Calculation					
Block 1				0.46	0.43
Intercept	75.62	4.59			
Age	22.29	5.08	0.61***		
Working Memory	4.97	5.08	0.14		
Block 2				0.17	0.60
Intercept	75.62	3.85			
Age	15.10	4.65	0.42**		
Working Memory	−0.93	4.53	−0.03		
Patterning	18.64	4.83	0.51***		
Predicting Concepts Scores					
Block 1				0.32	0.28
Intercept	27.78	5.14			
Age	18.21	5.69	0.50**		
Working Memory	4.63	5.69	0.13		
Block 2				0.00	0.26
Intercept	27.78	5.21			
Age	17.26	6.29	0.48**		
Working Memory	3.85	6.12	0.11		
Patterning	2.46	6.54	0.07		

\*\*  $p < 0.01$ .

\*\*\*  $p < 0.001$ .

#### 4. Discussion

Patterns are often considered central to early mathematics learning; yet, the empirical evidence linking early pattern knowledge to mathematics performance is relatively sparse. In the current study, 36 children ranging in age from 5 to 13 years old completed a pattern extension task with three pattern types that varied in difficulty. They also completed three math tasks that allowed for separate measures of calculation skill and knowledge of concepts. Children were successful on the pattern extension task, though older children fared better than younger children, potentially due in part to their sophisticated approaches to the task that considered both dimensions of the pattern (shape and size). Importantly, performance on the pattern extension task was associated with calculation skill, but was not associated with knowledge of concepts, after controlling for age and verbal working memory. Results suggest that patterning may play a key role in the development of some aspects of early mathematics knowledge, but that this may not hold for all forms of mathematics knowledge.

The current findings contribute to a growing body of literature foregrounding the central role of patterning in the development of mathematics knowledge. Most theories of early mathematics learning focus predominantly on children's numeracy knowledge – their knowledge of whole numbers and number relationships (e.g., Geary, 2011; Jordan, Kaplan, Ramineni, & Locuniak, 2009; LeFevre et al., 2010; Purpura & Lonigan, 2013). However, recent evidence has begun to highlight the important role of patterning in early mathematics (Fyfe & Rittle-Johnson, 2016; Kidd et al., 2014; Papic et al., 2011; Rittle-Johnson et al., 2016; Warren & Cooper, 2007). In line with this emerging body of work, the current study provides empirical support for a link between patterning skill and mathematics knowledge. Moreover, we demonstrate this link after controlling for both age and working memory capacity.

Unique to this study was the attempt to relate patterning skill to different aspects of mathematics knowledge, rather than to mathematics achievement more globally. Some recent work has explored relations between patterning and different content strands; for example, Rittle-Johnson et al. (2016) used patterning skills at the beginning of formal schooling to predict algebra, geometry, and numeracy achievement in fifth grade. However, each of these content strands is comprised of many component concepts and skills. The current study is the first to evaluate relations between patterning skill and specific aspects of mathematics knowledge in early childhood. Specifically, patterning skill was predictive of children's calculation skill. However, there was no evidence that patterning was associated with knowledge of two key math concepts, equivalence and inversion.

What might explain this pattern of findings? It does *not* seem to be due to one task requiring greater developmental maturity or cognitive capacity than the other. These pattern-math relations held after controlling for individual differences in age and verbal working memory capacity. Rather, the difference may be due to how calculation skill and knowledge of concepts are acquired. Fluency with calculation develops over a long period of time and is supported by identifying and generalizing predictable sequences in objects and numbers. For example: (1) in the count sequence, the digits 0 through 9 repeat over and over again in each decade (twenties, thirties, forties, etc.), (2) any number plus one is equal to the next number in the count sequence, (3) nine plus any number is equal to ten plus that number minus one, (4) adding two even numbers together results in a different even number, and so forth. Extracting these patterns can support counting and fluency with basic calculations (e.g., as in an addition facts table). Thus, patterning likely predicts calculation skill because it supports the ability to gradually identify and describe these predictable sequences of numbers across a wide range of experiences over a long period of time.

In contrast, children's knowledge of equivalence and inversion was assessed by their use of strategies based on fundamental

principles. For mathematical equivalence problems, children's correct strategies were based on a relational understanding of the equal sign, that is, knowing that the equal sign means "the same as." For inversion problems, children's strategies were based on knowledge of the inversion principle for addition and subtraction, that is, knowing that addition and subtraction "undo" one another. To succeed at each type of problem, there is one critical chunk of conceptual knowledge that is required—and this knowledge chunk may be acquired at a single moment rather than incrementally over time. Indeed, performance on both tasks was largely bimodal – suggesting that children either "had" the relevant knowledge chunk or did not. We suggest that patterning skill may not predict knowledge of equivalence and inversion because these concepts rely on a single chunk of knowledge, rather than on the gradual and incremental extraction of predictable patterns.

Note, however, that our findings leave open the possibility that patterning may be related to other concepts or other measures of conceptual knowledge, particularly ones that may be based on identifying and generalizing regularities (e.g., noticing that the sum of two natural numbers is greater than either addend, see Prather & Alibali, 2009). Further, the relation between patterning and various math concepts may differ across development given that children's understanding of key math concepts differs across ages. Future work with a variety of tasks and concepts is needed to evaluate the relations between patterning and conceptual understanding.

In addition to documenting relations between patterning skill and mathematical knowledge, the current study provides insight into the development of children's pattern knowledge on a unique patterning task. Although pattern extension itself is widely used to assess children's patterning skill (e.g., Clements et al., 2008; Warren & Cooper, 2006), varying the pattern along two dimensions (shape and size) in different combinations allowed us to vary the difficulty of the task and make it appropriate for a wider age range (see also Rittle-Johnson, Saylor, & Swygart, 2008). This is the first study to examine children's performance on patterns in which unit length and unit type differed across dimensions (i.e., the different-different items).

Several key findings about the development of patterning skill emerged. First, children were quite successful on the pattern extension task, particularly on same-same and same-different items (which are similar to items used in previous research). Indeed, on average, children solved 90% of these items correctly, and even the younger group (ages 5–8) solved 85% of these items correctly. However, children were not at ceiling, particularly on the different-different items on which they had to attend to both dimensions. Average performance on these items was only 65%. Second, patterning success was also predicted by verbal working memory capacity, over and above the effects of age. This is consistent with previous work showing robust correlations between repeating pattern knowledge and individual differences in working memory capacity (e.g., Miller et al., 2016; Rittle-Johnson et al., 2013). Third, children offered a wide variety of explanations. As in prior work (Rittle-Johnson et al., 2015) the most common explanation was to label the elements in the pattern in order. The labels-both-dimensions explanation had a high success rate, was positively correlated with accuracy on the difficult different-different items, and older children were more likely to offer this explanation.

A final contribution of the current study is to provide insight into children's developing mathematics knowledge on three different tasks. Performance on the standard arithmetic problems varied widely by age, with older children (9–13) near ceiling and younger children (5–8) closer to 50% correct. Accuracy on the inversion problems was similar, but use of the shortcut strategy was much lower and bimodal. Age did not predict use of the shortcut strategy. Performance on the mathematical equivalence problems was also low and bimodal. Of those children who solved them correctly, most were from the older group. Younger children were more likely to resort to common, incorrect strategies, such as adding all the numbers in the problem (e.g., Alibali, 1999; McNeil & Alibali, 2005; McNeil, Fyfe, & Dunwiddie, 2015; Perry et al., 1988). Thus, on average, children's calculation skills were moderate and their knowledge of equivalence and inversion was low. Age was often, but not always, related to performance, and verbal working memory was not related to performance after controlling for age.

Although children's knowledge of both inversion and equivalence was relatively low, several pieces of evidence suggest that children tend to acquire knowledge of inversion earlier than knowledge of mathematical equivalence. First, more children used a conceptually sophisticated strategy on inversion problems than on mathematical equivalence problems. Sixty percent of children never used the inversion shortcut on the inversion problems, but a full 72% never used a conceptually correct strategy on the equivalence problems. Second, among children who exhibited knowledge of one concept but not the other, 70% had knowledge of inversion but not equivalence. Finally, children who displayed knowledge of inversion but not equivalence were much younger, on average, than children who displayed knowledge of equivalence but not inversion or children who displayed knowledge of both concepts. Thus, although both tasks have been well-studied individually (see Crooks & Alibali, 2014, for a review), this study provides some of the first evidence on how these two tasks are related within the same sample.

Despite the positive contributions of the current work, several limitations suggest directions for future research. First, our study provides additional correlational – but not causal – evidence in favor of relations between patterning and mathematics knowledge. Research using experimental designs is needed to test whether pattern interventions cause improvements in math performance. Second, although we controlled for age in our analyses, our sample size did not allow us to test for interactions of age and pattern skills. Thus, we could not examine whether the relations between patterning and mathematics vary with age. This is a critical issue for future research, given how widely math skills can vary across the age range included in this study.

Other limitations relate to the specific measures we employed. We assessed patterning skill using a single task – namely, children's ability to extend a repeating pattern. Future work should examine pattern-math relations using a wider range of tasks and pattern types. Rittle-Johnson et al. (2015) suggest that pattern abstraction – recreating a model pattern using novel materials – may be a more mathematically relevant task than extending an existing pattern. Other research suggests that skills for working with more complex pattern types (e.g., spatial and numeric growing patterns) may relate to performance in higher level mathematics. For example, Lee, Ng, Bull, Pe, and Ho (2011) found that children's skill with growing numeric patterns (e.g., 2, 4, 7, 8, 10) in elementary school was predictive of their algebra knowledge one year later. These complex patterns may tap more sophisticated abilities to detect and generalize predictable rules and may thus relate to other aspects of mathematics knowledge.

Our measures of children's mathematics knowledge also had several limitations. We relied on three very specific mathematics tasks with only a few items per task, we presented the tasks in a fixed order, and we used verbal self-report data to code children's strategy use. It is possible that children's verbal reports may have been inaccurate; for example, on the inversion problems, some children may have used conceptual strategies but believed that they "should" report formal calculation strategies. Children may also have used multiple strategies (e.g., initially solving using the shortcut and then "checking" via calculation), but reported only a single strategy. Future studies should use a wider range of measures of children's conceptual understanding, and should expand beyond self-reports of strategy use.

Finally, future work could also control for more and different cognitive skills using a larger sample. For example, we considered children's verbal working memory, but visual-spatial working memory is separable from verbal working memory and is also related to both math achievement (e.g., Bull, Espy, & Wiebe, 2008; McKenzie, Bull, & Gray, 2003) and patterning (e.g., Collins & Laski, 2015).

The current results shed light on the central role of patterning in early mathematics. They suggest that theories of mathematics development should go beyond numeracy and include knowledge of repeating patterns as a key component (see Rittle-Johnson et al., 2016). Further, they highlight the need to consider differing types of mathematics knowledge, as we found that patterning predicted calculation skill, but we did not find that patterning predicted important mathematics concepts. We suggest that early educators, including both parents and teachers, should consider teaching patterns both directly (e.g., having children create, extend, and abstract repeating patterns using blocks, beads, or crayons) and indirectly (e.g., organizing arithmetic practice in a way that highlights key predictable sequences in the numbers), but in a way that is integrated with other forms of mathematics knowledge. For although some claim that "patterns are what mathematics is all about" (Steen, 1988, p. 616), our findings suggest a more nuanced take – patterning skills are centrally important for some aspects of early mathematics knowledge.

## Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at <http://dx.doi.org/10.1016/j.cogdev.2017.07.003>.

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