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Variability in the natural number bias: Who, when, how, and why

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ABSTRACT

When reasoning about rational numbers, people sometimes incorrectly apply principles or rules for natural numbers. Many factors affect whether participants display this *natural number bias*, including their age and experience, the affordances and constraints of the given task, and even the specific numbers in the given problem. In this paper, we argue that this variability can be conceptualized in terms of dynamic choices among problem-solving strategies. People's strategy choices vary as a function of their repertoire of available strategies and as a function of the specifics of the tasks, problems, and context. Further, we argue that the specific profiles of variability in strategy use that are observed in different participant groups can be conceptualized in terms of the strength and precision of the representations of numbers and operations that people in those groups possess. In our view, the natural number bias arises when people's representations of rational number magnitudes or rational number operations are not sufficiently strongly activated or sufficiently precise to guide performance on a specific task in a specific context. In these cases, participants' more highly activated or more precise representations for natural numbers may underlie and guide their performance. This account suggests that contexts and experiences (including instructional experiences) that help build, strengthen, and activate rational number representations should lead to improvements in performance.

1. Introduction

People sometimes use principles or rules for natural numbers in situations where those principles or rules do not apply. For example, when comparing fraction magnitudes, people sometimes choose as larger the fraction that has larger whole numbers as parts; so, for example, people sometimes judge 1/4 as greater then 1/3, because 4 is greater than 3 (Behr, Wachsmuth, Post, & Lesh, 1984). Similarly, when comparing decimal magnitudes, people sometimes choose as larger the decimal that is longer; so, for example, people sometimes judge .835 as greater than .87, because 835 > 87 (Durkin & Rittle-Johnson, 2015). This tendency to over-extend natural, whole number principles and rules, particularly when reasoning about rational numbers, has been termed the *whole number bias* (Ni & Zhou, 2005), or, alternatively, the *natural number bias* (NNB).

People's behavior is inherently variable (Alibali & Sidney, in press; Siegler, 2007), and behavior that manifests the NNB is no exception. There is substantial variability in whether and how the NNB is manifested across people, tasks and contexts. Not all people display the NNB in every situation in which they could or might be expected to do so. Moreover, people sometimes display the NNB on some items for a given task, but not on others.

In this commentary, we first consider variability in the NNB. Put simply, *who* displays the NNB, and *when* do they do so? Next, we seek to explain this variability in terms of *how* people solve problems that involve rational numbers. Finally, we consider *why* people manifest the specific profile of variability that we observe. We argue that the nature of the observed variability depends on the representations that are most highly activated for a given person and in a given context. We close by considering the implications of this view for future research.

2. Variability in the natural number bias: who shows it, and when?

Who displays the natural number bias? It seem that everyone does, in some form or another. The NNB been observed in elementary school students (e.g., Meert, Grégoire, & Noël, 2010), in high school students (e.g., DeWolf & Vosniadou, 2015), in adults (e.g., Vamvakoussi, Van Dooren, & Verschaffel, 2012b), and even in expert mathematicians (e.g., Obersteiner, Van Dooren, Van Hoof, &

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Verschaffel, 2013). Thus, the bias is evident, not only in learners who have just been introduced to rational numbers, but also in individuals who have extensive familiarity with rational numbers. However, the pattern of tasks, contexts, and response measures on which the NNB is displayed varies across participant groups. In some cases, people display the NNB in their patterns of errors; in other cases, in reaction times; and in still other cases, in the strategies that they report using to solve problems. In our view, this variability is potentially informative about the processes that give rise to the NNB.

When do people display the NNB? That is, what sorts of tasks elicit thinking based on natural numbers? One commonly used task is fraction magnitude comparison. In such tasks, people are asked either to choose the larger (or smaller) of two fractions (e.g., Behr et al., 1984; Kallai & Tzelgov, 2009), to judge whether a statement about relative fraction magnitudes (e.g., 7/8 > 2/3) is true or false (e.g., Bonato, Fabbri, Umiltà, & Zorzi, 2007; Vamvakoussi et al., 2012b), or to compare the magnitude of a given fraction to a "standard" value (e.g., is 4/9 more or less than 3/5?) (e.g., DeWolf, Grounds, Bassok, & Holyoak, 2014; Schneider & Siegler, 2010). Response patterns for different types of comparisons indicate that participants sometimes rely on knowledge of the magnitudes of the component natural numbers (i.e., the numerators and denominators), rather than on the magnitudes of the fractions themselves. However, participants who display the NNB in fraction magnitude comparisons do not do so on every trial. For example, DeWolf and Vosniadou (2015) report that on some trials, adults rely on strategies based on natural number parts, leaving them susceptible to the NNB, but on other trials, they appear to rely on magnitude representations for fractions, and therefore do not display the NNB.

People also display the NNB when comparing decimal magnitudes, for example, when asked to identify which of two decimals is greater, or when asked to identify decimals that are "worth the same amount" as a target decimal (e.g., "Circle all the numbers that are worth the same amount as 0.51: 0.5100, 0.051, 0.510, 51"; Durkin & Rittle-Johnson, 2015). On items such as these, participants often respond in ways that reveal that they view decimals as having certain properties of natural numbers (e.g., longer numbers are greater, adding a zero to the end of a number increases its magnitude). However, participants do not display NNB-based responses on every trial. For example, Durkin and Rittle-Johnson reported that, on a pretest, 9–11 year old participants displayed the NNB on about half of the items on which they might be expected to do so.

Another task in which people display the NNB is in interpreting algebraic expressions that involve operations (e.g., Vamvakoussi, Van Dooren, & Verschaffel, 2012a). Van Hoof and colleagues (Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015) asked high school students to judge whether certain expressions (e.g., $x < x + 2$, $x > x * 2$, $3 < 3/x$) could or could not be true. They found that students almost always considered natural number properties when evaluating the algebraic expressions (i.e., by invoking principles about natural numbers, such as "multiplication makes larger" or by replacing the unknown with a natural number), regardless of the operation. In contrast, students considered other kinds of numbers (e.g., negative whole numbers) only when evaluating expressions with addition and subtraction operations, and not when evaluating expressions with multiplication and division operations. Thus, students applied NNB-based approaches more broadly than other approaches.

3. How does the NNB arise? A dynamic strategy choice account

Some researchers have argued that the NNB arises as a consequence of a dual processing system for magnitudes (e.g., Vamvakoussi, et al., 2012b). According to this view, people engage in both *intuitive* processing, in which knowledge about numerical magnitudes is immediately and automatically activated, and therefore guides performance, and in *analytic* processing, in which the magnitudes of component parts are accessed and combined in an explicit, effortful (though sometimes error-prone) way. From this perspective, the key distinction is between a fast, automatic system and a slower, effort-driven system for processing magnitudes. Errors that manifest the NNB occur when (1) natural number magnitudes are quickly and automatically activated, and (2) effortful, analytic processing does not override those automatically activated representations. Thus, "errors may be attributed to S1's [the intuitive system's] pervasiveness and S2's [the analytic system's] failure to intervene" (Vamvakoussi et al., 2012b, p. 347).

The dual-processing account provides an apt explanation for patterns of reaction times and errors on magnitude comparison tasks. However, it fares less well in explaining patterns of responses among students asked to interpret algebraic expressions (e.g., Van Hoof et al., 2015). Qualitative data suggest that in interpreting algebraic expressions, students apply a range of explicit strategies informed by natural number knowledge.

Building on this prior work, we propose that it may be useful to incorporate both intuitive and analytic processes in a *dynamic strategy choice* account. This account acknowledges the role of intuitive processes while focusing in greater detail on the nature of people's effortful processing. Our proposal emphasizes (1) variability in people's effortful approaches to solving problems (an issue that has received relatively little attention in previous work), and (2) systematic patterns of strategy use—including strategies based on intuitive processes and those based on effortful processes—across problems, contexts, and individuals. Importantly, we do not believe that all cases of NNB stem from intuitive processes (although some do), nor that all cases of effortful processing yield responses free of NNB (although some do).

We argue that people's strategy choices depend on their repertoires of available strategies. Further, drawing on concepts from dynamic systems theory (Smith & Thelen, 2003), we argue that an individual's strategy choices also depend on the affordances of particular problems, tasks, or contexts, and on that individual's history of solving similar and related problems. These factors combine to influence behavior (including strategy choices, reaction times, and errors) in the moment. Thus, patterns of strategy use, errors, and reaction times should reveal variability, but there should also be systematicity within that variability. From this perspective, we argue that responses that manifest the NNB arise as a result of people selecting problem-solving strategies based on natural numbers in certain types of problems or in certain contexts.

In our view, one class of strategies for solving magnitude comparison problems is based on intuitive processes (akin to *retrieval* as a strategy for solving arithmetic problems; Siegler & Shipley, 1995). Specifically (and cast in strategy choice terms), one possible strategy for solving fraction magnitude comparison problems is simply to compare magnitude representations that are automatically activated (i.e., by the intuitive processing system). For example, in comparing the magnitudes of ½ and ¼, people may "just know" that ½ is greater than ¼ — or that 4 is greater than 2—and they may choose to respond on the basis of this highly activated, intuitive knowledge, whether it be knowledge of fraction magnitudes or whole number magnitudes. Thus, intuitive processing can yield responses that manifest the NNB, or responses that do not.

Another class of strategies for solving magnitude comparison problems involves more effortful strategies (akin to "*back-up*" strategies for solving arithmetic problems; Siegler & Shipley, 1995). People may reach an answer by taking a series of steps that involve encoding and operating on specific features of the problems. For example, in fraction magnitude comparison tasks, participants may effortfully compare numerators and denominators, or they may convert fractions with unlike denominators to fractions with a common denominator, and then compare numerators. In some cases, these effortful strategies draw on natural number representations when it is inappropriate to do so; such strategies manifest the NNB.

In our view, the distinction between intuitive and analytic processing is orthogonal to the question of whether individuals draw on knowledge of natural numbers or knowledge of rational numbers in a given problem or context. Of course, early in development, intuitive reasoning is based largely on natural number knowledge, but as learners develop stronger representations of rational numbers, the intuitive processing system can draw on those representations, as well. Likewise, effortful, analytic processes are often applied to rational numbers, but they can be applied to natural numbers, as well. In our view, people's thinking about natural numbers is implicated, not only in strategies that rely on intuitive processing, but also in many types of effortful strategies. Thus, we agree with DeWolf and Vosniadou (2015) that "the intervention of whole number thinking in rational number tasks is not a unitary process that exhibits itself only in reaction time studies, but a very complex phenomenon" (p. 42). We argue that it is important to consider the intervention of natural number thinking in effortful strategy use, as well its role in intuitive strategy use.

We argue that people's strategy choices depend on many factors, including aspects of the given problem, the task context, and the individual's knowledge and history of solving related problems. Thus, in our view, many factors influence whether more effortful or more intuitive processes guide behavior at a given moment. Moreover, our account highlights that there are many different effortful strategies, and that intuitive and effortful approaches may combine in performance at a given moment. Note that we do not claim that people's choices among strategies are always conscious or explicit. In some cases, strategy choices may be explicit, and in other cases, they may emerge implicitly, as people engage with the presented problems (Siegler & Stern, 1998).

From a dynamic strategy choice perspective, a critical task for research and theory is to specify the nature of the strategies that people use over development, and to account for how strategies are deployed across problems, tasks, contexts, and participant groups. We argue that the affordances of particular tasks or particular problems may "push" people to apply specific strategies. For example, fraction comparisons with common numerators (e.g., 1/6 vs. 1/11) afford a simple and effective "compare denominators" strategy, whereas fraction comparisons without common components do not afford such strategies. Further, as people's skills and knowledge develop, they may acquire new strategies, or the way that they deploy strategies across problems, tasks, and contexts may shift.

For example, consider people's performance on fraction magnitude comparison tasks. For some individuals or participant groups, their strategy choices across problem types might be summarized in the following way: (1) if the fractions to be compared have common numerators or common denominators, then compare natural number parts that differ and report the result; (2) if the fractions do not have common numerators or common denominators, and natural number parts are congruent (i.e., both larger in one of the fractions) compare the natural number parts and report the result; (3) if the fractions do not have common numerators or common denominators, and natural number parts are *not* congruent, guess. For other individuals or participant groups, their strategy choices might be summarized differently. For example, the strategy choices of the expert mathematicians studied by Obersteiner et al. (2013) could be summarized as follows: (1) if the fractions to be compared have common numerators or common denominators, then compare natural number parts that differ and report the result; (2) if the fractions do *not* have common numerators or common denominators, access magnitudes, and if the difference in magnitudes is *large*, compare and report the result; (3) if the fractions do *not* have common numerators or common denominators, access magnitudes, and if the difference in magnitudes is *small*, convert the fractions to a common denominator, compare and report the result.

As a second example, consider people's performance on interpreting algebraic expressions. For some people, their strategy choices across expression types might be summarized in a single step: Test a natural number and report the result. For other people, the strategy choice process might be captured differently: (1) if the problem involves multiplication or division, test a natural number and report the result; (2) if the expression involves addition or subtraction, test a natural number, then test zero, then test a negative whole number, and then report the result.

Of course, these characterizations are also generalizations, as many other factors may also contribute to variability in performance, including individuals' knowledge about the specific numbers or operations involved in the problems, individuals' recent history with related problems, and moment-to-moment factors, such as attentional focus. Nevertheless, these systematic patterns of variability across problem types are revealing about the nature of people's knowledge, how it changes with age and experience, and how it is activated and used in performance in the moment.

4. *Why* do people display the observed profile of variability in the natural number bias?

In both of the foregoing examples, people seem to be making strategy choices, whether explicitly or implicitly, that maximize their efficiency and accuracy in reaching solutions. Thus far, however, we have not addressed what causes certain strategies to be more efficient or accurate for certain people in certain contexts. We hypothesize that the efficiency or accuracy of a particular strategy depends on the strength of activation of the mental representations needed to apply that strategy.

Let us first consider fraction magnitude comparison problems. For such problems, the efficiency of different strategies for a given individual presumably depends on the strength and precision of that individual's *magnitude representations* for the relevant natural numbers and fractions. These magnitude representations are presumably activation functions, with activation being highest for the actual magnitude of the number being represented and decreasing as the distance from that number increases (see, e.g., Dehaene, 2009; Nieder & Dehaene, 2009; Prather, 2012). More precise magnitude representations have higher peaks at the target magnitude and smaller variance in the activation functions. A peaked representation of magnitude accurately targets a specific magnitude; thus, it refers to what what DeWolf and Vosniadou (2015) term a "robust representation of fractions on the number line" (p. 42) that can be accessed as a unit.

Let us next consider problems that involve interpreting algebraic expressions. For such problems, the efficiency of different strategies for a given individual may depend on the strength and precision of that individual's *representations of the operations* involved. These representations may incorporate explicit knowledge about how those operations "work" (e.g., "addition makes larger") (Dixon, Deets, & Bangert, 2001), explicit knowledge about the relational structure of the operation across number types (e.g., “division of whole numbers and division of fractions both involve making groups as big as the divisor”) (Sidney & Alibali, 2014), or implicit knowledge of regularities that occur when the operation is applied (e.g., the sum of two natural numbers is larger than either addend) (Dixon & Bangert, 2005; Dixon et al., 2001; Prather & Alibali, 2009; 2011). We refer to this knowledge as knowledge of *operational patterns*, and we note that such knowledge can be specific to particular types of numbers (see, e.g., Prather & Alibali, 2008, for evidence regarding negative numbers).

We propose that the strength of activation and the precision of people's mental representations, both of magnitudes and of operations, vary depending on a number of factors, including experience with the relevant external representations (e.g., for fraction magnitude comparisons, the amount of experience with fractions), the recency of previous activation of the relevant mental representations (e.g., for fraction magnitude comparisons, how recently a given fraction was previously encountered or considered), and task affordances that might support accessing the mental representation (e.g., for fraction magnitude comparisons, the presence of a number line with ¼, ½, and ¾ marked).

In addition, magnitude representations for all numbers become stronger and more precise over developmental time, resulting in parallel developments in magnitude representations for natural and rational numbers, though at different timescales (Siegler, Thompson, & Schneider, 2011; Torbeyns, Schneider, Xin, & Siegler, 2015). Representations of operations on natural and rational numbers presumably also become stronger and more precise over developmental time; however, empirical evidence on this point is fairly limited (though see Dixon & Bangert, 2005; Dixon et al., 2001; Sidney, Chan, & Alibali, 2013).

We argue that mental representations that are more highly activated—either by having high baseline activation due to familiarity or by being highly activated due to recency or due to affordances of the task context—are more likely to be used in reasoning. When mental representations for rational numbers are highly activated, participants will draw on that knowledge more or less directly (what deWolf and Vosniadou refer to as “accessing representations directly as a unit”, without being influenced by natural number components), and use strategies that utilize those representations. However, in contexts in which mental representations for rational numbers are not highly activated, participants will tend to use strategies that draw on other representations that are more highly activated, typically those for natural numbers. People may utilize those representations either directly or in more effortful ways. Thus, this proposed account makes predictions about when people will rely on rational number representations and when they will rely instead on mental representations of natural numbers. The account also highlights that once activated, those mental representations can be used in reasoning in a range of ways, both effortful and not.

The papers in this special issue suggest at least two ways in which the strength and precision of people's mental representations of numbers and operations affect their likelihood of successfully drawing on those representations in reasoning: (1) by directly supporting reasoning about those numbers and operations, and (2) by influencing whether those representations are activated when people are asked to think about numbers or operations, more generally. We consider each of these ideas in turn.

First, we propose that people's mental representations of numbers and operations directly support reasoning about those numbers and operations. For magnitude comparison problems, representations of specific numbers and their magnitudes are most relevant; the strength and precision of these representations increase over time and with experience, as people become more familiar with number symbols and the magnitudes they refer to. For interpreting algebraic expressions, representations of operations and their effects on specific sorts of numbers are most relevant; the strength and precision of these representations presumably increase over time and with experience, as people practice applying operations and observing their effects.

Some research suggests that, as the strength and precision of people's representations of particular numbers improves, people become better able to reason about *properties* of those numbers. McMullen, Laakkonen, Hannula-Sormunen, and Lehtinen (2015) found that understanding of rational number magnitudes is a prerequisite for understanding the density of rational numbers. Their findings suggest that students require strong, precise magnitude representations in order to successfully reason about the relationships among several magnitudes, and in turn about continuity and density. Importantly, McMullen and colleagues also found that students' magnitude representations themselves have a delayed and likely difficult path to mastery; specifically, they noted that rational number instruction takes “time to sink in” (p. 19). One way to think about such knowledge “sinking in” is in terms of increases in the strength and precision (and consequently the accessibility) of the students' representations of rational numbers, which presumably take place as the students engage in extended practice with rational numbers in a variety of contexts. Changes in these representations may also play a role in conceptual change more generally.

Second, we propose that the strength and precision of people's mental representations of numbers and operations influence whether those representations are activated when people are asked to think about numbers or operations, more generally. For example, when asked to reason about *numbers*, the types of numbers that people call to mind will depend on the nature of their mental representations of specific numbers. In other words, what “counts” as a number itself undergoes conceptual change, as mental representations within each number category become stronger and more precise. A similar phenomenon would presumably occur if people were asked to reason about operations, more generally, as well, although we know of no data that directly address this issue.

Two of the papers in this special issue present findings relevant to this idea. Durkin and Rittle-Johnson (2015) studied change over time in students' activation of knowledge about different categories of numbers. They found that students moved from using whole number principles as the basis for evaluating decimal number magnitudes to using principles consistent with fractions. Thus, when asked to judge decimal magnitudes, students initially relied on their strongest, most highly activated representations of numerical magnitude – those for natural numbers. With instruction about decimals, students came to represent decimals more strongly as numbers, they became more likely to draw on principles associated with other kinds of numbers, as well. Siegler and colleagues have proposed that the integration of rational numbers into the more general category of numbers—and thus the ability to recognize fundamental aspects of all numbers—underlies the strong relationship that is observed between knowledge of fraction magnitudes and general mathematics achievement (Siegler et al., 2011; Siegler & Lortie-Forgues, 2014; Torbeyns et al., 2015).

The phenomenon described by Van Hoof and colleagues (2015) follows a similar pattern. When asked to reason about algebraic expressions in a general way, participants of all ages drew on their most highly activated representations—those for operations on natural numbers. In the case of multiplicative operations, students tended to rely *solely* on their knowledge of operations on natural numbers. In the case of additive operations, some students also considered operations on negative whole numbers, as well. This could be viewed as evidence that students' representations of additive operations were becoming sufficiently robust to incorporate operational patterns for negative integers. In contrast, few students considered how those operations apply to rational numbers, implying that students' representations of both additive and multiplicative operations were not yet robust enough to incorporate operational patterns for rational numbers.

The level of activation strength or precision needed in order for a representation to be used in a given task may also vary with specifics of the task or context. For example, in accordance with previous research on fraction magnitude comparisons (e.g., Schneider & Siegler, 2010; DeWolf & Vosniadou, 2015) found that when magnitude differences were great, participants were more likely to make quick decisions, which were presumably based on accessing magnitudes. However, when magnitude differences were small, participants were slower, suggesting that they used more time-consuming, effortful strategies that rely on natural number components. When distances are great, participants' internal representations for fraction magnitudes need not be particularly precise in order to yield accurate judgments. However, when distances are small, participants appear to be sensitive (implicitly or explicitly) to the potential pitfalls of relying on their imprecise magnitude representations directly in making judgments.

In brief, we suggest that the natural number bias appears when people's representations of rational number magnitudes or rational number operational patterns are not sufficiently precise or sufficiently highly activated to guide performance on specific tasks in specific contexts. In these cases, representations of natural number magnitudes or natural number operational patterns, which are stronger and more precise by virtue of being earlier learned and more frequently activated, tend to dictate performance.

This account opens up new questions about what makes a mental representation sufficiently strong or precise, and about the consequences of supporting activation through features of the task context or through recent experience. Specifically, this account implies that strengthening or sharpening students' mental representations for rational numbers—by increasing their familiarity, by activating them more recently, and by strengthening their connections to other kinds of numbers —should affect whether or not students are susceptible to the natural number bias.

In future research, one might investigate ways in which various manipulations of these factors would affect individuals' likelihood of demonstrating the NNB. For example, in the case of fraction magnitude comparisons, participants may more readily rely on magnitude representations for pairs of fractions for which magnitude representations are both strongly activated. Indeed, this view suggests that one could alter participants' patterns of strategy use on magnitude comparison tasks by giving them practice with specific fractions. After such practice, comparisons that involve practiced items should be solved using different strategies than comparisons involving less well-practiced items. For example, if one group of children received extensive practice with magnitudes 1/5, 1/8, and 1/11, and another group with 1/6, 1/9, and 1/12, one might expect different patterns of strategy use in children's performance on fraction magnitude comparisons involving those fractions.

5. Conclusion

The body of work presented in this special issue highlights variability in people's thinking about rational numbers. Many factors affect whether participants display the natural number bias, including their age and experience, the affordances and constraints of the given task, and even the specific numbers in the given problem. We have argued here that it may be useful to conceptualize variability in performance in terms of a dynamic strategy choice account, which argues that people deploy different strategies as a function of their repertoires of available strategies and as a function of the specifics of the problems, tasks and contexts. Further, we argue that the specific profiles of variability in strategy use that are observed in different participant groups can be conceptualized in terms of variations in the strength and precision of the representations of numbers and number concepts that people in those groups possess. We suggest that the NNB arises when people's representations of rational number magnitudes or rational number operational patterns are not sufficiently strongly activated or sufficiently precise to guide performance on a specific task in a specific context. In these cases, participants' stronger, more precise, and more highly activated natural number representations may underlie and guide their performance. Thus, this account suggests that contexts and experiences (including instructional experiences) that help build, strengthen, and activate rational number representations may lead to improvements in performance.

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