



Special issue paper

Perceptual support promotes strategy generation: Evidence from equation solving

Martha W. Alibali^{1*} , Noelle M. Crooks² and Nicole M. McNeil³

¹University of Wisconsin, Madison, Wisconsin, USA

²Broward College, Fort Lauderdale, Florida, USA

³University of Notre Dame, Notre Dame, Indiana, USA

Over time, children shift from using less optimal strategies for solving mathematics problems to using better ones. But why do children generate new strategies? We argue that they do so when they begin to encode problems more accurately; therefore, we hypothesized that perceptual support for correct encoding would foster strategy generation. Fourth-grade students solved mathematical equivalence problems (e.g., $3 + 4 + 5 = 3 + \underline{\quad}$) in a pre-test. They were then randomly assigned to one of three perceptual support conditions or to a Control condition. Participants in all conditions completed three mathematical equivalence problems with feedback about correctness. Participants in the experimental conditions received perceptual support (i.e., highlighting in red ink) for accurately encoding the equal sign, the right side of the equation, or the numbers that could be added to obtain the correct solution. Following this intervention, participants completed a problem-solving post-test. Among participants who solved the problems incorrectly at pre-test, those who received perceptual support for correctly encoding the equal sign were more likely to generate new, correct strategies for solving the problems than were those who received feedback only. Thus, perceptual support for accurate encoding of a key problem feature promoted generation of new, correct strategies.

Statement of Contribution

What is already known on this subject?

- With age and experience, children shift to using more effective strategies for solving math problems.
- Problem encoding also improves with age and experience.

What the present study adds?

- Support for encoding the equal sign led children to generate correct strategies for solving equations.
- Improvements in problem encoding are one source of new strategies.

Children often solve mathematics problems incorrectly or inefficiently, and their performance improves over time and with experience. These improvements are often due to changes in the strategies children use to solve problems (e.g., Lemaire & Callies, 2009; Torbeyns, De Smedt, Ghesquiere, & Verschaffel, 2009). With development,

*Correspondence should be addressed to Martha W. Alibali, University of Wisconsin, 1202 W. Johnson St., Madison, WI, USA (email: mwalibali@wisc.edu).

children generate or adopt newer and better problem-solving strategies and abandon old, less-effective ones. What factors motivate this change? And how does this change occur? Answering these questions is at the heart of understanding cognitive development and learning.

In this research, we investigate the role of perceptual factors in children's strategy change in mathematics. A growing body of work highlights the importance of perceptual factors in shaping cognitive performance, even in domains, such as mathematics, that are considered 'abstract' (e.g., Goldstone, Landy, & Son, 2010). In this research, we investigate whether children generate new strategies for solving mathematics problems when they receive perceptual support for accurately encoding the problems. We define *encoding* as internally representing features of the presented problem, and we define *strategies* as sets of actions that can be taken to arrive at solutions to problems. Children may construct or select a strategy based on the features of the problem that they have encoded. From this perspective, then, improvements in problem encoding might enable children to generate new strategies by highlighting new problem features that they can use.

Abundant evidence from a range of problem domains indicates that both problem encoding and strategy use improve with age, experience, and expertise (e.g., Booth & Davenport, 2013; Dean & Malik, 1986; Feil & Mestre, 2010; Fujimura, 2001; Siegler & Chen, 2008; Werner & Thies, 2000). Moreover, several lines of evidence suggest that changes in encoding and changes in strategy use are causally related. First, improvements in encoding are associated with improvements in performance on a range of tasks (e.g., McNeil & Alibali, 2005; Rittle-Johnson, Siegler, & Alibali, 2001). Second, manipulations that 'trick' people into using incorrect strategies are effective, in part, because they lead people to misencode the problems (Crooks & Alibali, 2013).

Most critically, experimental manipulations intended to promote accurate problem encoding lead to improvements in problem solving. As one example, Siegler (1976) noted that young children often fail to accurately encode critical features of balance-scale problems – problems in which weights can be placed on pegs on two sides of a fulcrum. Many children encode only the number of weights, and not their distance from the fulcrum. When children were instructed how to encode both weight and distance appropriately, many progressed from using an incorrect, weight-only strategy to using a strategy that takes both weight and distance into account (Siegler, 1976).

Along similar lines, Alibali, McNeil, and Perrott (1998) encouraged improved encoding in fourth-grade students learning to solve mathematical equivalence problems (e.g., $3 + 4 + 5 = 3 + \underline{\quad}$), which are arithmetic equations with operations on both sides of the equal sign. Some students were directed to 'Notice the equal sign', and others were not. Students rarely expressed new strategies in speech, but those who were directed to 'Notice the equal sign' often generated new strategies that they expressed solely in gestures. That is, following the encoding manipulation, students were more likely to produce responses in which they expressed one strategy in speech, and a different strategy in the accompanying gestures (termed 'gesture-speech mismatch' responses; see Perry, Church, & Goldin-Meadow, 1988).¹

In both of these studies, the encoding instructions were direct and explicit – in each case, children were directed to attend to specific problem features. In everyday settings, however, children seldom receive direct instruction about what to encode. However,

¹ For evidence that such gestured responses reflect strategy knowledge, see Garber, Alibali, and Goldin-Meadow (1998).

instructional materials and others' actions may provide implicit support for accurate encoding by highlighting critical features of problems in various ways. Several studies have suggested that perceptual support for accurate encoding, even without direct instruction about what to encode, can facilitate learning and problem solving (e.g., DeLoache, 1986; Jamet, Gavota, & Quaireau, 2008; Joh & Spivey, 2012). Moreover, *misleading* perceptual cues can also hinder people's problem solving (Jiang, Cooper, & Alibali, 2014; Landy & Goldstone, 2010). These lines of work highlight the importance of implicit, perceptual factors in guiding problem encoding.

The aim of this study was to examine whether perceptual support for accurate encoding, in the absence of direct instruction about what to encode, would lead elementary school students to generate new, correct strategies for solving mathematical equations. We investigated this issue among fourth-grade students learning to solve mathematical equivalence problems. Elementary school students in the United States and Canada often encode such problems incorrectly and use incorrect strategies to solve them (e.g., Matthews & Rittle-Johnson, 2009; McNeil, 2007; Sherman & Bisanz, 2009). Here, we focus on students' generation of new, correct strategies for such problems.

Students may be more likely to use encoded problem features to generate strategies when they realize that their existing strategies are incorrect. When they know they are incorrect, they may shift to using another strategy drawn from their existing repertoire or they may attempt to generate a new strategy. If, at the same time, students begin to encode features of problems that they had previously not encoded, they may attempt to use those features in constructing a new strategy. From this perspective, the combination of feedback about correctness and perceptual support for accurate encoding should create optimal conditions for strategy generation. On this basis, we hypothesized that perceptual support for correct encoding, in the context of feedback that existing strategies are incorrect, would promote generation of new, correct strategies.

In designing our perceptual support manipulations, we were guided by past research on the features of equivalence problems that students frequently misencode. Two such features are the equal sign and the right side of the problem (McNeil & Alibali, 2004). We hypothesized that perceptual support for encoding would be most beneficial if it targeted those features. Therefore, we designed our perceptual manipulations to do just that: One condition provided support for encoding the presence and position of the equal sign, and another provided support for encoding the structure of the right side of the problem (see McNeil & Alibali, 2004). In a third experimental condition, we provided support for encoding the numbers used in a common correct strategy, the grouping strategy (i.e., 'add the numbers that do not appear on both sides of the equation'). Because students typically encode the numbers in the equations correctly, we did not expect that students in this condition would improve their encoding of the problems. However, we expected that, by highlighting the two numbers that could be added to obtain the correct answer, along with the blank, we might encourage some students to generate the grouping strategy. In this case, perceptual support might increase activation on specific features that students encoded accurately even without support (i.e., those specific numbers and the blank), and this might make students especially likely to use those features in their strategies.

To provide perceptual support, we made the target features perceptually salient by printing them in a contrasting colour (red). Contrasting colour can draw attention by making a stimulus 'pop out' from its surroundings (Wolfe, 1998). Visual contrast can signal that information is important, and it may therefore encourage learners to encode that information (Joh & Spivey, 2012; Ozcelik, Arslan-Ari, & Cagiltay, 2010; Ponce &

Mayer, 2014). Thus, colour offered a straightforward way to support students' encoding of specific problem features.

Method

Participants

Participants were 113 fourth-grade students (56 females, 57 males) from parochial schools. Because our goal was to test whether improvements in encoding of equivalence problems would promote generation of correct strategies, we excluded participants who used correct strategies at pre-test ($N = 35$) or who reconstructed all of the equivalence problems correctly on the encoding pre-test ($N = 14$). One participant was excluded because he did not take the transfer test, and one was excluded due to experimenter error. The final sample consisted of 62 participants (35 females, 27 males), with a mean age of 9 years, 10 months (range 8 years, 10 months to 11 years, 2 months).

Assessments

To assess *problem-solving performance* and *strategy use*, we asked participants to solve equivalence problems presented one at a time on a white board, and to explain their solutions. For the problem-solving pre-test, post-test, and follow-up test, participants solved six equivalence problems (three with the blank immediately following the equal sign, e.g., $4 + 3 + 6 = _ + 6$, and three with the blank in final position, e.g., $3 + 9 + 5 = 3 + _$) and two traditional arithmetic problems (e.g., $5 + 6 + 4 + 5 = _$). During the intervention segment, participants solved three equivalence problems. In all equivalence problems, one number from the left side was repeated on the right side. No feedback was provided during pre-test, post-test, or follow-up test; however, feedback was provided to all participants with the intervention problems.

To assess *transfer*, we asked participants to complete a paper-and-pencil transfer test consisting of eight problems. Two problems were identical in form to those on pre-test and post-test, so these problems were not included in participants' transfer scores. The remaining six problems included two each of three types of transfer problems: addition equivalence problems without a repeated addend (e.g., $7 + 3 + 8 = _ + 6$), multiplication equivalence problems with a repeated multiplicand (e.g., $4 \times 5 \times 3 = _ \times 3$), and multiplication equivalence problems without a repeated multiplicand (e.g., $2 \times 4 \times 3 = _ \times 6$).

To assess *encoding* of equivalence problems, we used a problem reconstruction task, as in past research on equivalence (McNeil & Alibali, 2004) and in other domains (Chase & Simon, 1973; Siegler, 1976). Participants were presented with five problems, one at a time, each for 5 s. After each problem, participants were asked to write exactly what they saw on a note card. One problem was a traditional arithmetic problem; the other four were equivalence problems.

Procedure

Children participated individually in two videotaped sessions conducted in a quiet room at their schools. Sessions were scheduled approximately 4 weeks apart ($M = 28.5$ days, range 7–35 days). The average number of days between sessions was comparable across conditions, $F(3, 58) = 0.374, p = .77$.

Session 1

Participants first completed the problem-solving pre-test and encoding pre-test. Next, participants received a brief intervention, during which they solved three blank-final equivalence problems. For the intervention, participants were randomly assigned to one of four conditions: (1) Control, in which the problems were presented in black ink ($n = 18$ in the final sample); (2) Equal Sign, in which the equal signs in the problems were printed in red ink ($n = 15$ in the final sample); (3) Right Side, in which the right sides of the problems were printed in red ink ($n = 15$ in the final sample); and (4) Numbers, in which the two addends that would be grouped to obtain the correct answer for each problem, as well as the plus sign between those numbers and the blank, were printed in red ink ($n = 14$ in the final sample). Note that all three perceptual support conditions involved simply printing parts of the equations in red; in no case did participants receive explicit instruction about what to encode, about a problem-solving strategy, or about the meaning of the equal sign. For the three intervention problems, participants in all conditions (including the Control condition) received feedback about whether their strategies were correct. When participants used correct strategies, they were told, 'Great! That is a correct way to solve the problem'. When participants used incorrect strategies, they were told, 'That's a really good try, but that's not the correct way to solve the problem'. Participants were not given specific information about their errors. Following the intervention, participants completed the problem-solving post-test, the encoding post-test, and the transfer test.

Session 2

During the second, follow-up session, participants completed a problem-solving test and an encoding test.

Coding

Solution strategies

Participants' strategies for solving equivalence problems were coded using a system based on that developed by Perry *et al.* (1988), summarized in Table 1. Note that different incorrect strategies lead to distinct solutions. For each problem, the solution strategy was defined as the strategy leading to the solution, or if the solution was ambiguous, the strategy expressed in speech. When speech included deictic terms (e.g., 'I added these'), we also considered participants' gestures. To establish reliability, a subset of responses was coded by a second coder; agreement was 95% ($N = 190$).

Table 1. Strategies for solving $3 + 4 + 5 = 3 + _$

Strategy	Solution	Sample verbal explanation
Incorrect strategies		
Add-all	15	'I added 3 plus 4 plus 5 plus 3'
Add to equal	12	'I added 3 plus 4 plus 5 and that equals 12'
Carry	3	'There was a 3 here, so I put a 3'
Correct strategies		
Equalize	9	'3 plus 4 plus 5 equals 12 and 3 plus 9 equals 12'
Add-subtract	9	'I added 3 plus 4 plus 5 and then took away 3 from that'
Grouping	9	'I saw the two 3s, so I just added 4 plus 5, and that's 9'

We identified all of the strategies each participant used on the equivalence problems at each assessment point, and we compared the sets of strategies that each participant used at pre-test and at each of the later time points (intervention, post-test, follow-up). As in past research (Alibali, 1999), any strategy that a participant did not use at pre-test, but used at a later time point, was considered to be *generated* at the later time point.

Strategies were classified as either correct or incorrect (see Table 1). Total number of problems solved with correct strategies was computed for pre-test, post-test, and follow-up test (range 0–6 at each test).

Problem encoding

Participants' reconstructions of equivalence problems were coded using the system developed by McNeil and Alibali (2004). Reconstructions were coded as either correct or incorrect. For incorrect reconstructions, the type of error was also coded. *Number* errors involved misencoding the numbers or their order (e.g., reconstructing $3 + 4 + 5 = 3 + _$ as $3 + 7 + 5 = 3 + _$). *Conceptual* errors involved misencoding the problem structure (e.g., reconstructing $3 + 4 + 5 = 3 + _$ as $3 + 4 + 5 + 3 = _$, $3 + 4 + 5 + 3$, or $3 + 4 + 5 = + 3 _$). As in past work (e.g., Alibali, Phillips, & Fischer, 2009), one point was awarded for each reconstruction free of conceptual errors (range 0–4 at each test).

Transfer

Participants' solutions to the transfer problems were scored as correct or incorrect.

Results

We first consider changes in participants' encoding, and we then consider strategy generation and transfer. Conclusions are unchanged if participants' age is included in the statistical models as a covariate.

Did perceptual support lead to improvements in encoding?

We first asked whether perceptual support led to improvements in performance on the encoding tests. We expected that the two conditions that provided perceptual support for encoding features that students often misencode – the Equal Sign and Right Side conditions – would lead to the greatest improvements. Because students typically encode the numbers correctly at pre-test, we did not anticipate that the Numbers condition would lead to substantial improvements in encoding.

As seen in Figure 1, participants in each of the intervention conditions improved their encoding more across tests than participants in the Control condition. To evaluate these patterns statistically, we analysed the data using linear mixed-effects models (Bates, Maechler, Bolker, & Walker, 2014). The statistical model included condition and test as fixed effects; participant, item, and the slope of test within participants as random effects; and successful encoding of equivalence problems as the outcome measure.² Planned

²For this analysis, we used the following statistical model: `glmer(Score ~ Condition * Test + (1 + Test | Participant) + (1 | Item), family=binomial(link=logit))`.

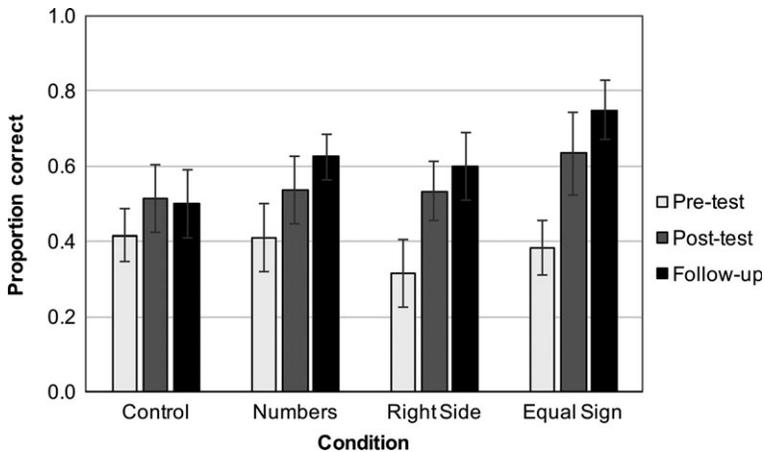


Figure 1. Average proportion of equivalence problems encoded correctly (out of 4) at each test as a function of condition. Error bars represent standard errors.

comparisons revealed that improvement in encoding from pre-test to follow-up was significantly greater in the Equal Sign condition than in the Control condition, $\beta = 1.84$, $z = 2.84$, $p = .004$. There was also a trend that improvement from pre-test to follow-up was greater in the Red Right Side condition than in the Control condition, $\beta = 1.16$, $z = 1.87$, $p = .06$. Improvement from pre-test to follow-up did not differ between the Numbers condition and the Control condition, $\beta = 0.66$, $z = 1.09$, $p = .28$. Thus, there was evidence that the red equal sign manipulation led to improvements in encoding, but there was not comparable evidence for the other manipulations. There may have been insufficient power to detect differences, given the substantial variability in encoding scores.

Did perceptual support foster generation of new strategies?

We next examined whether participants generated new strategies. Figure 2 presents the proportion of participants in each condition who generated strategies during the intervention problems, at post-test, and at follow-up test; the total height of the bars indicates the proportion who generated strategies of any kind, and the shaded portion indicates the proportion who generated correct strategies.

We used logistic regression to analyse whether the likelihood of generating a correct strategy depended on condition. We analysed the data for each time point (i.e., intervention, post-test, and follow-up) separately, because including time point as a within-subjects factor resulted in convergence issues or extremely large standard errors, presumably due to very low levels of correct strategy generation in some conditions at each time point.³ We used planned contrasts to compare strategy generation in each of the experimental conditions to the Control condition.

The pattern was the same at each time point: Relative to participants in the Control condition, participants in the Equal Sign condition were more likely to generate correct strategies; however, this was not the case for participants in the other two perceptual

³ For these analyses, we used statistical models of the form: `glm(GenerateCorrect ~ Condition, family = binomial(link = logit))`.

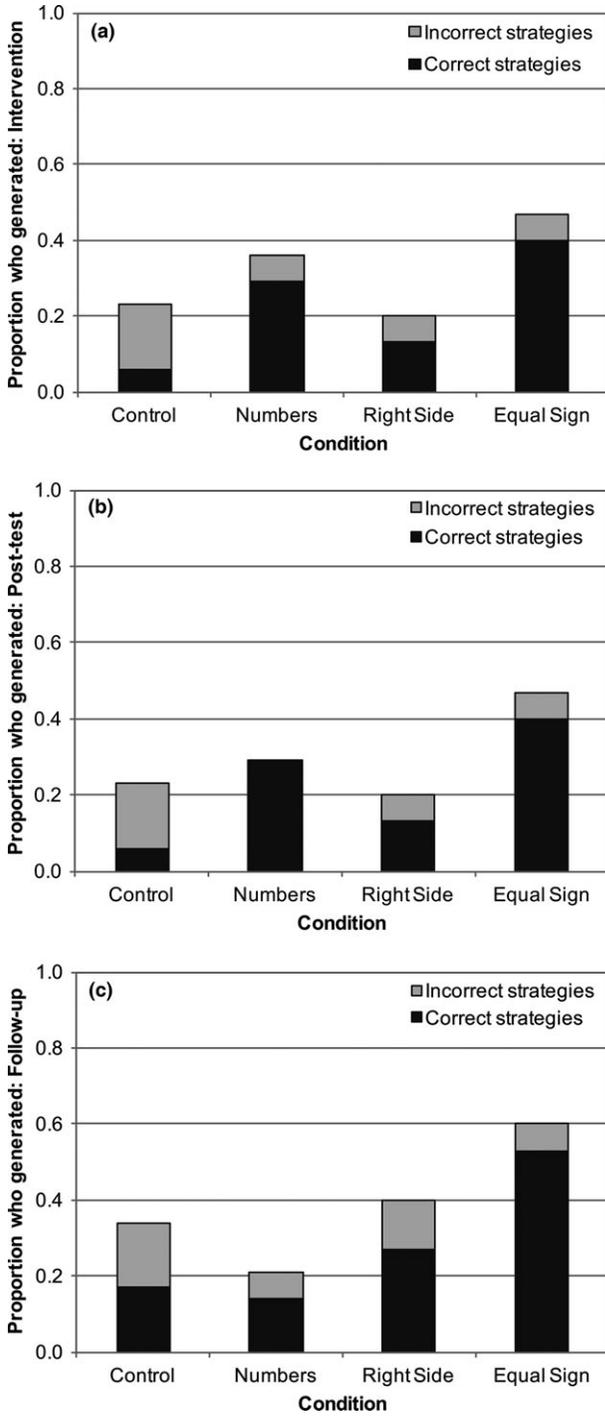


Figure 2. Proportion of children who generated correct strategies (black) and new incorrect strategies (grey) as a function of condition, during the intervention (Panel a), at post-test (Panel b) and at follow-up (Panel c).

support conditions. During the *intervention*, participants in the Equal Sign condition were more likely to generate new, correct strategies than Control participants, $\beta = 2.43, z = 2.10, p = .04$, but this was not the case for participants in the Right Side condition, $\beta = 0.96, z = 0.75, p = .45$, or participants in the Numbers condition, $\beta = 1.92, z = 1.62, p = .11$. The effect size was large; the odds of using a correct strategy during the intervention were estimated to be 11.33 times higher, 95% CI (1.18, 109.25), in the Equal Sign condition than in the Control condition. At *post-test*, participants in the Equal Sign condition were also more likely to generate correct strategies than Control participants, $\beta = 2.43, z = 2.10, p = .04$, but again, this was not the case for participants in the Right Side condition, $\beta = 0.96, z = .75, p = .45$, or participants in the Numbers condition, $\beta = 1.92, z = 1.62, p = .11$. Once again, the odds of using a correct strategy at post-test were estimated to be 11.33 times higher, 95% CI (1.18, 109.25), in the Red Equal Sign condition than in the Control condition. Finally, at *follow-up*, participants in the Red Equal Sign condition were again more likely to generate correct strategies than Control participants, $\beta = 1.74, z = 2.13, p = .03$, but this was not the case for participants in the Right Side condition, $\beta = 0.60, z = 0.70, p = .49$, or participants in the Numbers condition, $\beta = -0.18, z = -0.18, p = .85$. The odds of using a correct strategy at follow-up were estimated to be 5.71 times higher, 95% CI (1.15, 28.35), in the Equal Sign condition than in the Control condition.

These patterns of strategy generation resulted in superior performance on the problem-solving tests for participants in the Equal Sign condition; these data are presented in Figure 3. Because performance at each test was largely bimodal, we analysed the data using logistic regression; we defined a child as succeeding on each test if they used correct strategies on at least five (of six) equivalence problems.⁴ Across tests, participants in the Equal Sign condition were more likely to succeed than those in the Control condition, $\beta = 3.84, z = 2.55, p = .01$; however, this was not the case for participants in the other experimental conditions (Right Side, $\beta = 0.48, z = -0.29, p = .78$; Numbers, $\beta = -1.30, z = -0.52, p = .60$).

Did perceptual support promote transfer?

We also investigated whether perceptual support for accurate encoding fostered transfer to novel problem types (see Table 2).⁵ As seen in the table, for each problem type, levels of success were very low overall; however, on each measure, the proportion of children who succeeded was greatest in the Equal Sign condition.

Discussion

In this study, a simple perceptual manipulation altered participants' problem encoding and provoked strategy generation. When children received feedback that their existing strategies were incorrect, those who received perceptual support for correctly encoding the equal sign were more likely to generate new, correct strategies for solving the problems than those who did not receive support for encoding.

⁴For this analysis, we used the following statistical model: `glmer(Success ~ Test + Condition + (1|Participant), family = binomial(link = logit), with test centred`. Including the interaction of test and condition did not significantly improve model fit (tested via comparison of nested models), so we used the simpler model.

⁵For these analyses, we used statistical models of the form `glm(Success ~ Condition, family = binomial(link = logit))`.

Of course, our findings may hinge on some of the specifics of our experimental design. For example, perceptual support may be particularly effective when children are also asked to explain their problem-solving strategies, as we did in this study. Self-explanations are often beneficial for learning (for a review, see Rittle-Johnson, Loehr, & Durkin, 2017), and they may have potentiated the effects of perceptual support in the current design. Second, mathematical equivalence, as a domain, may have unique features that make perceptual support particularly effective, and these features may not hold in other domains. Children commonly display very specific deficits in encoding mathematical equivalence problems (e.g., McNeil & Alibali, 2004); therefore, perceptual support may be especially well suited to address the challenges children face in learning mathematical equivalence. This may not be the case for other topics, such as fractions or commutativity.

We hypothesize that perceptual support may be particularly beneficial for learners who are relatively ‘close’ to correct performance, as they may benefit from fairly minimal forms of scaffolding. Indeed, children show a wide range of responsiveness to lessons

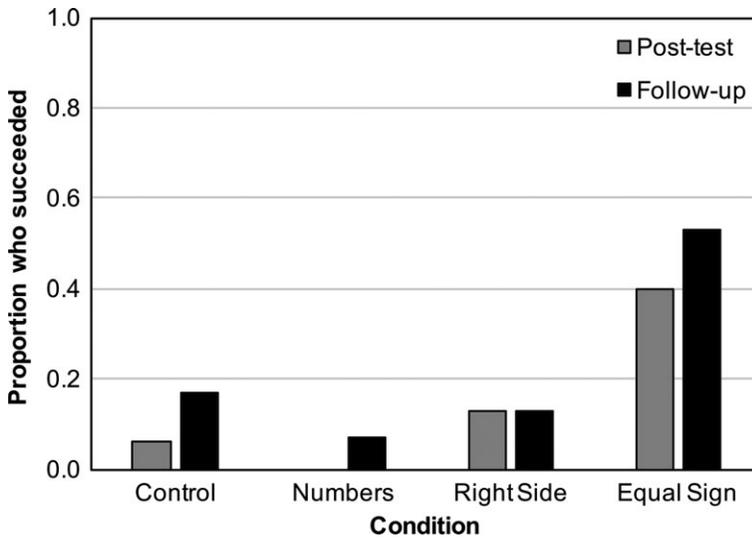


Figure 3. Proportion of children who succeeded on the post-test and follow-up test (i.e., who solved at least 5 of 6 problems using correct strategies) as a function of condition. Note that all children solved all problems incorrectly at pre-test.

Table 2. Percentage of children in each condition who solved at least 1 (of 2) of each type of transfer problem correctly, and who solved any (of 6) transfer problems correctly on the transfer test at the end of Session I

Problem type	Condition			
	Control	Numbers	Right side	Equal sign
Addition without equivalent addends	5.6	21.4	6.7	33.3*
Multiplication with equivalent multiplicands	5.6	21.4	13.3	26.7
Multiplication without equivalent addends	5.6	0	6.7	26.7
Any (of 6)	5.6	28.6	13.3	33.3*

Note. * $p < .10$, relative to the Control condition.

about mathematical equivalence, with some children displaying sudden, dramatic improvements after relatively minimal instruction, and others showing more limited improvements, even after many weeks of instruction (e.g., Alibali, 1999; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; McNeil, Fyfe, Petersen, Dunwiddie, & Brletic-Shipley, 2011; Perry, 1991). Perceptual support may be most beneficial for those students who already display a 'readiness' to learn (Perry *et al.*, 1988). We suggest that children who encode the equal sign inconsistently or inefficiently may be highly likely to profit from perceptual support, because they already display an emergent ability to encode the location of the equal sign. In contrast, children who fail to encode the equal sign altogether may need stronger forms of support, such as perceptual support combined with direct verbal instruction. More generally, children with different kinds of encoding deficits may need different levels of support to improve their encoding to a level that allows strategy generation.

We suggest that perceptual manipulations that remediate specific, relevant deficits in encoding may be particularly effective. In this study, we found consistently strong effects of the red equal sign manipulation, which we argue remediated students' inadequate encoding of the equal sign (either their inconsistent encoding or their failure to encode it altogether). Accurate encoding of the equal sign is incompatible with the most common incorrect approach to solving the problems – the *add-all* strategy – so it may make children especially likely to abandon *add-all* and generate something different. However, note that we did not directly test whether the red equal sign promoted generation of correct strategies *more so* than the other perceptual support manipulations. We suggest that the red right side manipulation may have partially remediated another, relevant deficit in encoding, namely inaccurate encoding of the right side of the equation. The red numbers manipulation may have cued a specific, correct strategy (i.e., the grouping strategy, 'add the numbers that do not appear on both sides'). However, overall levels of strategy generation in the Numbers and the Right Side conditions were not significantly greater than in the Control condition.

Unlike students in the Equal Sign condition, some students in the Numbers condition who generated strategies did not maintain those new strategies to the follow-up test – perhaps because those strategies were based on transient heightened activation of a problem feature, rather than lasting improvements in encoding accuracy. In support of this interpretation, there was also no evidence that students in the Numbers condition improved their problem encoding significantly in response to the manipulation. Students tended to encode the numbers well, even before the intervention, so the specific form of encoding support offered in the Numbers condition was not especially beneficial.

Indeed, perceptual support that does not lead to improvements in encoding may not be beneficial for problem solving. In a related study, a similar red equal sign manipulation was used with a different group of participants, and they did not improve their encoding and also did not generate correct strategies (Alibali, 2015). The participants displayed extremely poor encoding of the equations at pre-test (much poorer than participants in the current study) and the data suggest that the manipulation did not effectively remediate their substantial deficits in encoding. In this regard, however, it is important to bear in mind that the primary purpose of the present study was not to endorse colour as a form of perceptual support, but rather to test the hypothesis that strategy generation occurs when perceptual support leads participants to improve their encoding.

The present findings have implications for understanding the origins of new forms of behaviour – an issue that has long been recognized as a central challenge in cognitive development. In this study, perceptual support helped children react to feedback by

generating new, correct strategies. This finding suggests that there may be two key ingredients necessary for strategy generation to occur: a vulnerability to change, which can be indexed by perceptual readiness, and a ‘trigger’ that actually provokes change. In the present study, the red equal sign manipulation may have made participants perceptually ready to generate new strategies, and feedback served as the trigger for participants to actually do so. In the Control condition, in which participants experienced the trigger without being perceptually ‘ready’, generation was much less likely. We did not include a condition in which we encouraged perceptual readiness without also including a (feedback) trigger; however, past work suggests that such a condition is not likely to lead to new strategies manifested in problem solutions or speech (through it may encourage new strategies expressed in gesture; Alibali *et al.*, 1998). More generally, we hypothesize that perceptual readiness creates vulnerability to change, and once this vulnerability is in place, an external trigger for change will result in strategy generation. Note that this view is similar to ‘diathesis-stress’ models of the development of psychopathology (e.g., Gazelle & Ladd, 2003; Walker & Diforio, 1997; Zuckerman, 1999). Future research is needed to test whether this framework provides a good model for understanding strategy generation.

Of course, in arguing that improved encoding is a source of new strategies, we have not solved the problem of the genesis of new forms – we have simply pushed the problem down a level, highlighting the need to understand the origins of improvements in encoding. Nevertheless, we believe that the present findings do represent progress, because they identify directions for future work about mechanisms of strategy discovery. We suggest that at least some solutions to the puzzle of strategy discovery may lie in perceptual features of the environment, in experiences that highlight particular perceptual features, and in perceptual learning processes.

The present findings are also of interest as a demonstration that learning at one level of the cognitive system can provoke changes at other levels. The notion of interactions across levels is a central theme of many developmental systems theories (e.g., Gottlieb, 2007), and different levels are sometimes conceptualized in terms of the differing timescales at which activities occur (Newell, 1990). Importantly, changes at lower levels of processing and briefer timescales (e.g., perceiving individual equations) can cumulate to have consequences at higher levels and longer timescales (e.g., equation-solving performance and strategy use, as in this study). These changes in turn may continue to ‘snowball’ to the even higher level and longer timescale of educational outcomes (see Anderson, 2002; Nathan & Alibali, 2010).

Recent evidence supports this view, specifically for mathematical equivalence. Middle-school students’ errors in using the equal sign (e.g., moving or deleting the equal sign) are associated with poorer performance on end-of-year achievement tests, and have been deemed a ‘troublesome sign for students’ math achievement’ (Booth, Barbieri, Eyer, & Paré-Blagoev, 2014; p. 19). Thus, instructional support for encoding equations – specifically, for encoding the equal sign – may contribute positively to students’ later math achievement (but see Booth & Davenport, 2013, for an alternative view).

Our findings also add to the growing body of work on perceptual processes in students’ interpretations of mathematical inscriptions (e.g., Jiang *et al.*, 2014; Landy & Goldstone, 2007a, 2007b, 2010; Massey, Kellman, Roth, & Burke, 2011). Specifically, they converge with other research suggesting that colour can be an effective way to guide learners’ attention to key features of instructional materials (Kalyuga, Chandler, & Sweller, 1999; Ozcelik, Karakus, Kursun, & Cagiltay, 2009; Ozcelik *et al.*, 2010).

In this research, we manipulated perceptual salience experimentally; however, in children's everyday experience, many kinds of experiences may affect perceptual salience. A teacher may use gesture or contrasting colour to guide attention to critical problem features, or may present related problems side-by-side to highlight contrasting features (Alibali, Nathan, & Fujimori, 2011; Lobato, Ellis, & Munoz, 2003). A parent may line up puzzle pieces with straight sides to emphasize their common feature (Levine, Ratliff, Huttenlocher, & Cannon, 2012). Children may also improve their encoding via perceptual learning processes (e.g., Kellman, Massey, & Son, 2009) that occur as they encounter problems in their everyday experience, both in school and out. Our findings suggest that basic perceptual mechanisms, such as visual pop-out and perceptual learning, may deliver features to the cognitive system that can then be used for strategy generation.

Once acquired, newly encoded features are available as raw material for constructing new strategies; we argue that having this raw material makes children vulnerable to change. Our findings suggest that, to help students learn in optimal ways, educators should identify essential perceptual features of instructional materials and provide students with experiences that support their encoding of those key features. Along similar lines, to make scientific progress on the question of how strategy discovery occurs, researchers should focus on identifying factors that provoke changes in learners' perceptual encoding of problems.

Acknowledgements

We thank Leslie Petasis, Tiffany Husby, Georgia Seeley, Amy Foster, and Michelle Benjamin for their assistance in collecting data, coding, and establishing reliability, and we thank Andrew Young and Pooja Sidney for assistance with data analysis and interpretation. This research was supported by a grant from the National Science Foundation (BCS-0096129) to Martha W. Alibali.

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Received 2 December 2016; revised version received 5 June 2017