MECHANISMS OF CHANGE IN THE DEVELOPMENT OF MATHEMATICAL REASONING

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I. Introduction

How do children progress from less knowledge to more knowledge in a domain? Understanding this transition is at the heart of understanding all forms of knowledge change, including changes due to development, learning, and instruction. To deeply understand change, we must be able not only to characterize the shape of change, but also to explain how change occurs. In other words, we must be able to specify the mechanisms that give rise to change. Detailed knowledge about the mechanisms of change will allow us to predict who will change and when, enable us to set up conditions to foster or inhibit change, and guide us about how to intervene when change does not occur as desired.

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Note that I refer here to *mechanisms* of change, in the plural rather than the singular. It is virtually certain that multiple change mechanisms apply in every domain. Different mechanisms may be responsible for changes observed in different individuals, at different times, and with respect to different specific content. The particular mechanisms at work in any particular individual's development depend on many factors, including characteristics of that individual and the individual's opportunities for learning, both formal and informal.

In this chapter, I focus on change mechanisms that apply in the development of mathematical reasoning. Mathematical reasoning is used in a variety of situations, including both formal mathematical tasks and everyday activities, such as shopping, managing finances, and cooking (e.g., doubling recipes, estimating quantities), so it is an important type of reasoning to understand in its own right. In addition, mathematical reasoning is a useful domain within which to study change mechanisms, because many changes in people's mathematical thinking and problem solving can be readily observed, and because such changes are amenable to experimental investigation. Furthermore, understanding change in the domain of mathematical reasoning may prove useful for designing optimal instruction and for understanding mathematics learning difficulties.

I focus largely on mathematical reasoning in the elementary and middle school years, although the mechanisms discussed are neither limited to these periods, nor limited to the domain of mathematics *per se*. Indeed, the same mechanisms may apply to other types of content (e.g., scientific reasoning, statistical reasoning) and with respect to earlier achievements, such as the development of counting, and later achievements, such as the acquisition of key concepts in calculus. However, most of the mechanisms discussed in this chapter have been studied most intensively in school-age children.

The purpose of this chapter is to review research regarding two classes of change mechanisms that have been proposed to apply in the domain of mathematical reasoning: (1) mechanisms that involve reciprocal relations between knowledge of problem-solving procedures and knowledge of concepts, and (2) mechanisms that involve expressing knowledge in speech and gestures. Of course, these two classes of mechanisms are not the only ones that have been proposed to apply in the development of mathematical reasoning. The present review is intended to be focused rather than exhaustive.

Mechanisms of change can be characterized at different grain sizes, and they may operate at different levels, ranging from the behavioral level (e.g., self-explaining) to the neural level (e.g., strengthening or weakening of neural connections). The mechanisms that are the focus of the current chapter are characterized at the behavioral level. For each class of mechanisms, I begin with global claims and proceed to more detailed hypotheses about the functioning of specific mechanisms.

II. Mechanisms that Involve Reciprocal Relations between Knowledge of Problem-Solving Procedures and Knowledge of Concepts

Mathematical knowledge consists of several different types of knowledge (e.g., Hiebert & LeFevre, 1986; Bisanz & LeFevre, 1990). Among these are knowledge of concepts (e.g., the principle that the two sides of an equation represent the same quantity) and knowledge of procedures for solving problems (e.g., procedures for isolating variables in algebraic equations). As many researchers have acknowledged, defining conceptual and procedural knowledge precisely is difficult, as is distinguishing them in practice. For present purposes, I adopt the definitions used by Rittle-Johnson and Alibali (1999). *Conceptual knowledge* involves understanding of principles that govern a domain and of relations among pieces of knowledge within a domain. *Procedural knowledge* involves knowledge of problem-solving procedures, or action sequences for solving problems.

Past research suggests that conceptual and procedural knowledge are intertwined in multiple and complex ways (e.g., Carpenter, 1986; Rittle-Johnson, Siegler, & Alibali, 2001; Canobi, Reeve, & Pattison, 2003). Indeed, gains in each type of knowledge can lead to gains in the other (Rittle-Johnson & Alibali, 1999). Thus, one class of mechanisms of change in mathematical reasoning involves reciprocal relations between knowledge of concepts and knowledge of problem-solving procedures. In the following subsections, I review evidence about how knowledge of concepts and knowledge of procedures influence one another. In each subsection, I first consider evidence that one type of knowledge may influence the other, and then consider potential mechanisms that may underlie that influence.

A. CONCEPTS INFORM PROCEDURE GENERATION

Several sources of evidence converge to suggest that gains in conceptual knowledge can influence procedural knowledge. First, a number of studies in various mathematical domains have shown that instruction that focuses on conceptual principles leads students to generate new problem-solving procedures. These include studies of decimal fractions (e.g., Hiebert & Wearne, 1989), multi-digit arithmetic (e.g., Fuson & Briars, 1990; Hiebert & Wearne, 1996; Blöte, Van der Burg, & Klein, 2001), and mathematical equivalence (e.g., Perry, 1991; Alibali, 1999). As one example, Rittle-Johnson and Alibali (1999) examined changes in the procedures that fourth- and fifth-grade students used to solve mathematical equivalence problems, which are equations with addends on both sides of the equal sign, such as $3 + 4 + 6 = 3 + \_$. Some of the students received a brief lesson focusing on the principle that the two sides of an equation represent the same quantity. Most students who received this lesson generated correct procedures for solving the problems after instruction, including procedures such as "find the sum of the left side, and then find a number that makes the sum on the right side the same as that on the left" (the *equalize* procedure) and "cancel the number that is the same on both sides, and add the remaining numbers" (the *cancel-and-group* procedure).

Second, some studies have shown that children with greater conceptual knowledge display greater gains in procedural knowledge after instruction. For example, Rittle-Johnson, Siegler, and Alibali (2001) assessed children's conceptual understanding of decimal fractions before and after an intervention that included a brief lesson. They also assessed children's procedural skill at placing decimal fractions on the number line before, during, and after the intervention. Children who had higher scores on the conceptual knowledge pretest made greater improvements in procedural knowledge from the pretest to the later segments of the study. Thus, amount of initial conceptual knowledge was associated with size of gains in procedural knowledge.

These studies raise the question of *how* conceptual knowledge leads to changes in procedural skill. By what processes might conceptual knowledge inform procedure generation? In considering this issue, Baroody (2003) distinguished between direct and indirect effects of conceptual knowledge on procedural advances. The direct pathway involves *conceptual instigation* of procedural innovations, such that innovations are directly motivated or triggered by new conceptual knowledge. The indirect pathway involves *conceptual support* for procedural advances that are motivated by other, non-conceptual factors. A wide variety of non-conceptual factors may compel children to change their problem-solving procedures, including a desire for cognitive economy, feedback about solution correctness, and outside intervention (e.g., seeing an adult or another child model a different procedure). The distinction between conceptual instigation and conceptual support is useful to bear in mind when considering the mechanisms that may underlie the effects of conceptual knowledge on procedural knowledge.

Conceptual knowledge may influence improvements in procedural knowledge through a number of possible mechanisms, either by directly instigating procedural innovations, or by supporting procedural advances driven by other, non-conceptual factors. In the following sections, I consider three potential mechanisms, and review existing empirical evidence for each: (1) gains in conceptual knowledge may lead to changes in problem representation, which in turn may enable the generation of new procedures; (2) gains in conceptual knowledge may lead children to realize that existing procedures are incorrect, and this may trigger procedure generation; and (3) gains in conceptual knowledge may guide children's evaluation of potential, alternative procedures, whether those procedures are self-generated or learned from outside sources.

1. Changes in Problem Representation

One direct pathway by which gains in conceptual knowledge may instigate procedural innovations is by causing changes in problem representation, which in turn may enable procedural innovations. Problem representation can be defined as "the internal depiction or recreation of a problem in working memory during problem solving" (Rittle-Johnson, Siegler & Alibali, 2001, p. 348). A new problem representation is created each time a problem is solved. Problem representations are sometimes inaccurate or incomplete because solvers may fail to represent certain problem features or may represent them inaccurately.

Past research from a variety of domains supports each of the links in the pathway from gains in conceptual knowledge to improved problem representation to gains in procedural knowledge. Problem solvers who have greater conceptual knowledge in a domain tend to form more accurate and more complete problem representations (e.g., Chi, Feltovich, & Glaser, 1981, physics problem solving), and experimental manipulations designed to improve solvers' problem representations lead to generation of new problem-solving procedures (e.g., Siegler, 1976, balance scale problem solving). Within mathematics, evidence from two domains, decimal fractions and mathematical equivalence, supports both of the links in this hypothesized pathway.

First, consider the evidence from the domain of decimal fractions. As described previously, Rittle-Johnson, Siegler, and Alibali (2001) assessed children's procedural skill at placing decimal fractions on a number line on a pretest, during an instructional intervention, and on a posttest. They also assessed children's conceptual understanding of decimal fractions at pretest and posttest. Rittle-Johnson *et al.* found that pretest to posttest gains in children's conceptual knowledge of decimal fractions were associated with improvements in their abilities to correctly place a decimal fraction on a number line.

Crucially, this relation was accounted for, in part, by children's representations of the decimal fractions, which were assessed based on explanations that the children provided during the intervention. On each of the 12 intervention trials, children solved a problem (e.g., indicated the location on a number line corresponding to a given decimal fraction), were told the correct answer, and then were asked to explain why that correct answer was correct. For each explanation, children were scored as having a correct representation if they conveyed either a *common-unit* understanding of the fraction, in which the fraction was represented in terms of its smallest unit (e.g., 0.625 is represented as 625 thousandths), or a *composite* understanding of the fraction, in which the fraction is represented as the sum of the individual column values (e.g., 0.625 is represented as the sum of 6 tenths, 2 hundredths, and 5 thousandths). One common incorrect representation of the fraction stemmed from an analogy to whole numbers (e.g., 0.625 is represented as "in the six hundreds").

Using mediation analyses, Rittle-Johnson *et al.* demonstrated that (1) conceptual knowledge at pretest was associated with frequency of correct problem representation during the intervention, (2) frequency of correct problem representation during the intervention was associated with gains in procedural knowledge from pretest to posttest and transfer test, and (3) the relation between initial conceptual knowledge and gains in procedural knowledge was diminished (although not completely eliminated) when frequency of correct problem representation was included in the regression model. Thus, problem representation accounted for at least part of the relation between initial conceptual knowledge and procedural knowledge gain. Children with greater conceptual knowledge were more likely to accurately represent the decimal fractions, and children who had better representations of the fractions displayed better learning of how to place the fractions on the number line.

The components of this pathway have also received empirical support in the domain of mathematical equivalence. Several studies have shown that instructional interventions that are designed to inculcate conceptual knowledge of mathematical equivalence also lead to improvements in problem representation. In one study (McNeil & Alibali, in press-b), children in a conceptual-instruction group received a brief intervention about the meaning of the equal sign, and they were also shown a correctly solved equivalence problem. Children in a control group were shown a correctly solved equivalence problem but received no other intervention. Children's problem representations were assessed both before and after the interventions by asking them to reconstruct equivalence problems that they viewed for 5 seconds each. As seen in Figure 1, among children who represented equivalence problems poorly on the pretest, children in the conceptual intervention condition made greater improvements in problem representation than did children in the control condition. In another study, Rittle-Johnson and Alibali (1999) provided children with a lesson that focused on the goal of making both sides of the problem equal, but that did not provide any guidance about how to achieve this goal. Among children who represented equivalence problems incorrectly on a pretest assessment, 58% of the children who received the conceptual lesson represented the problems correctly on a posttest assessment, compared to only 30% of children in a control condition who did not receive instruction. Thus, manipulations designed to promote conceptual knowledge also lead to gains in problem representation.

Evidence from studies of mathematical equivalence has also shown that improvements in problem representation can lead to generation of new procedures for solving equivalence problems. Alibali, McNeil, and Perrott (1998) presented some children with equivalence problems in which the equal sign was printed in red, and also told them to "notice where the equal sign

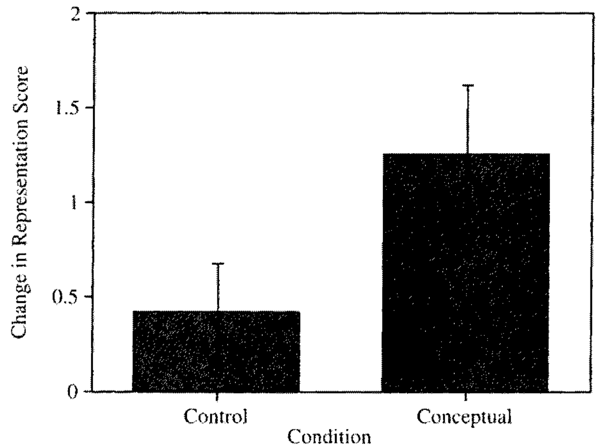

Fig. 1. Mean pretest to posttest improvements in representation scores for children who received a brief conceptual lesson and children who did not. Children who performed well on the pretest (i.e., who reconstructed at least two of three problems correctly) were excluded from the analysis. Children in the conceptual condition ($N = 12$) made greater improvements in problem representation than did children in the control condition ($N = 12$), $t(22) = 1.84$, $p < 0.05$. Data drawn from the study reported in McNeil and Alibali (in press-b).

is in the problem." This intervention was intended to highlight the position of the equal sign, which children often fail to represent correctly (McNeil & Alibali, in press-b). Children who received this intervention were more likely than children who did not to generate new procedures for solving the problems on a posttest. However, children expressed most of their new procedures in gestures and not in speech on the posttest. These new gestured procedures tended to occur in responses in which children expressed one (old) procedure in speech and a different (new) procedure in the accompanying gesture.[1] For example, as seen in Table I (third row), a child might express the *add-all* procedure in speech, but at the same time, express the *equalize* procedure (i.e., make both sides equal) in gesture. Responses in which speech conveys one procedure and gesture conveys another procedure have been termed "gesture–speech mismatches" in prior work (Perry, Church, & Goldin-Meadow, 1988). Furthermore, the procedures expressed uniquely in gesture in such mismatches have been shown in other research to reflect implicit, emerging procedures (e.g., Garber, Alibali, & Goldin-Meadow, 1998). Thus, these data suggest that changes in representation helped "get the ball rolling" in the process of generating new procedures.

[1] For details about how the system for coding gestures was developed, see Perry, Church, and Goldin-Meadow (1988).

TABLE I

Sample Gesture–Speech Match and Mismatch Responses for Mathematical Equivalence Problems

	Verbal explanation	Gestured explanation
	Gesture–speech matches	
1	I added 3 plus 9 plus 5 plus 3, and I got 20. (*add-all*)	Right hand point: left 3, 9, 5, right 3, solution. (*add-all*)
2	3 plus 9 plus 5 is 17, and 3 plus 14 is 17. (*equalize*)	Left hand point: sweep under 3 + 9 + 5. Right hand point: sweep under 3 + _. (*equalize*)
	Gesture–speech mismatches	
3	3 plus 9 is 12, plus 5 is 17, plus 3 makes 20. (*add-all*)	Left hand point: sweep under 3 + 9, 9 + 5. Right hand point: sweep under 3 + _. (*equalize*)
4	I added 3 and 9 and 5 and got 17, and 3 plus 14 is 17. (*equalize*)	Right hand point: 9, 5, solution. (*group*)

All examples are for the problem 3 + 9 + 5 = 3 + _. Procedure codes are indicated in parentheses.

Note: Procedures in gesture and speech are coded independently, using systems developed by Perry, Church, and Goldin-Meadow (1988). For each response, the codes assigned to the verbal and gestured explanations are compared. If the same procedure is assigned to the verbal and gestured explanations, the response as a whole is considered a gesture–speech match. If different procedures are assigned to the verbal and gestured explanations, the response as a whole is considered a gesture–speech mismatch.

In a follow-up study (Alibali & McNeil, in preparation), children were again presented with equivalence problems in which the equal sign was printed in red, and they also received feedback that the procedures they had used to solve the pretest problems were incorrect. Compared to the initial study in which children did not receive such feedback, many more children in this study generated new procedures that they expressed in speech. In the red-equal-sign condition, nearly half of the children expressed new procedures in speech on the posttest. Moreover, most of the new procedures that children generated involved the equal sign in some way. The most commonly generated procedure was the *equalize* procedure (a correct procedure), and the second most commonly generated procedure was the *add-to-equal-sign* procedure (an incorrect procedure), both of which rely on representing the position of the equal sign.

Taken together, these two studies suggest that changes in representation are a key component of the process of procedure generation. When children notice new features of problems, but do not realize that their existing procedures are incorrect, they generate new procedures and express them in gesture.

When children receive feedback that their existing procedures are incorrect, they generate new procedures and express them in speech. Thus, changes in problem representation promote procedure generation. When children begin to represent new problem features, they become able to generate new procedures that depend on those features.

From this perspective, one can think more concretely about the processes by which gains in conceptual understanding lead to generation of new procedures. Imagine a fourth-grade girl who lacks a sophisticated concept of the equal sign, and who instead thinks that the equal sign means "put the answer." Many late elementary and middle school students hold this view (Kieran, 1981; Rittle-Johnson & Alibali, 1999; McNeil & Alibali, in press-a). When presented with an equivalence problem such as $3 + 4 + 5 = 3 + \_$, the girl is likely to ignore the position of the equal sign, and instead represent the problem as a typical addition problem, such as $3 + 4 + 5 + 3 = \_$ (McNeil & Alibali, 2004). She is also likely to solve the equivalence problem incorrectly, using a procedure such as *add-all*, and arriving at a solution of 15.

Suppose this girl then learns via instruction that the equal sign does not mean "put the answer" but instead means that the quantities on each side of it are the same. If she is then presented with an equivalence problem, she might begin to notice the equal sign in the problem and, for the first time, begin to represent the two "sides" of the problem. Once the "sides" of the problem are represented as problem features, a whole new set of procedures become possible. The girl may then generate the *equalize* procedure, which involves making both sides equal.

2. Realizing that Existing Procedures Are Incorrect

Another direct pathway by which gains in conceptual knowledge may instigate procedural innovations is via the realization that existing procedures are incorrect. If acquiring new conceptual knowledge leads children to realize that their existing procedures are faulty, they may be compelled to construct new procedures to "fill the gap." In this case, conceptual knowledge may both instigate construction of a new procedure and support children's efforts to construct a new procedure that is correct.

An example may clarify the processes involved in this pathway. Consider the girl in the previous example, who learned that the equal sign means that the quantities on either side of it are the same. The girl may attempt to link this new conceptual knowledge with her existing procedure for solving equivalence problems, namely, the *add-all* procedure. However, in doing so, she may realize that her existing procedure is incompatible with this meaning of the equal sign. This may lead her to conclude that her existing procedure is incorrect, and compel her to generate a new one.

In constructing a new procedure, the girl may also utilize her newly acquired conceptual knowledge to guide her representation of the problem, and she may begin to represent features of the problem that she had heretofore represented incorrectly or not represented at all, such as the position of the equal sign. Thus, realizing that her existing procedure is incorrect may compel her to change her representation of the problem, engaging the mechanism discussed previously.

Although this pathway from new conceptual knowledge to realizing an existing procedure is incorrect to generating a new procedure is a plausible one, to my knowledge, there is no empirical data to support the entire pathway. However, there is ample evidence for the second link in the pathway, namely that children change their procedures when they learn that their existing approaches are incorrect. In one of the experimental conditions in Alibali (1999), children were asked to solve equivalence problems in an intervention, and for problems that were solved incorrectly, children were told, "That's a good try, but it's not the right way to solve the problem." Thirty-one percent of the children in this condition generated new procedures for solving the problem on a posttest, compared to no children in a control condition in which children did not receive feedback about the correctness of their prior approach.

Some data can also be marshaled to support the first link in the pathway, namely, the link from gains in conceptual knowledge to realizing an existing procedure is incorrect. In Rittle-Johnson and Alibali's (1999) study of mathematical equivalence, children evaluated three correct and three incorrect procedures both before and after a brief conceptual lesson, or without instruction in a control group. All of the children used incorrect procedures at pretest. The data of interest in the present context are children's evaluations of the *add-all* procedure, the most common incorrect procedure for problems of the form used in the study ($a + b + c = a + \_$). Children rated the procedures on a scale from "not so smart" (0) to "very smart" (2). Ratings of the *add-all* procedure given by children in the conceptual-instruction group decreased substantially (from 1.32 to 0.53), whereas ratings of children in the no-instruction control group held steady (from 1.22 to 1.28). These data suggest that the conceptual instruction led children to re-evaluate the *add-all* procedure, which the majority of the children used at pretest, and realize that this procedure was not correct.

Although these data are suggestive, it would be more definitive to directly assess children's acquisition of the target concepts, and to examine the relation between acquisition of the concepts and evaluation of pretest procedures. If this pathway works as hypothesized, children who acquire the target concepts should evaluate their incorrect pretest procedures more negatively at posttest than children who do not acquire the target concepts. A reanalysis of data from Rittle-Johnson and Alibali (1999) supports this hypothesis. Children were classified as having gained conceptual knowledge if they scored higher on the posttest conceptual knowledge assessment than on the pretest conceptual knowledge assessment. As seen in Figure 2, children who gained conceptual

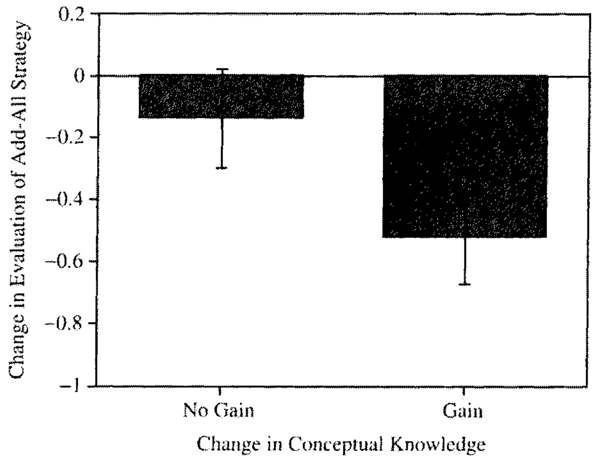

Fig. 2. Mean pretest to posttest changes in children's evaluations of the add-all *procedure for children who made gains in conceptual knowledge from pretest to posttest and children who did not make gains, collapsing across experimental conditions. Children who made gains in conceptual knowledge decreased their evaluations of the* add-all *procedure more than did children who did not make gains in conceptual knowledge, $t(46) = 1.71$, $p < 0.05$, one-tailed. Data drawn from Rittle-Johnson and Alibali (1999).*

knowledge over the course of the study decreased their evaluations of the *add-all* procedure more than did children who did not gain conceptual knowledge.

It would also be of value to address the entire pathway in a single study—ideally a microgenetic study that would examine conceptual knowledge, procedure evaluation, and procedure use at multiple time points. Such a study could address the temporal relations between gains in conceptual knowledge and changes in procedure evaluation.

3. Guiding Evaluations of Alternative Procedures

A third pathway by which conceptual knowledge may influence procedural knowledge is that gains in conceptual knowledge may guide children's evaluation of potential alternative procedures. Children may invent new procedures themselves, or they may learn new procedures from a variety of outside sources, such as observing other children, reading textbooks, learning from instruction, and so forth. Conceptual knowledge may inform children's decisions about whether to adopt these procedures, and may guide their evaluation of procedures invented to meet the demands of the current problem. Thus, in this pathway, conceptual knowledge is a support for procedural advances that are motivated by other, non-conceptual factors, such as modeling, instruction, or a desire for efficiency.

This pathway was first described by Siegler and Crowley (1994), who argued that children use conceptual knowledge, in the form of "goal sketches," to evaluate alternative procedures for solving problems. Goal sketches incorporate information about the goals that procedures within a domain must meet. In an experiment on children's procedures for solving simple addition problems (e.g., $4 + 8 = ?$), Siegler and Crowley found that 5-year-olds who did not yet spontaneously use the *count-from-larger-addend* procedure judged it to be smarter than an equally novel illegitimate procedure, and just as smart as their typical procedure of counting from one. The illegitimate procedure involved counting one of the addends twice, so it was inconsistent with the hypothesized goal sketch for simple addition. Thus, conceptual knowledge, in the form of a goal sketch, appeared to guide children's evaluation of the novel procedures.

Other findings support the idea that gains in conceptual knowledge influence children's evaluations of alternative procedures. As noted earlier, in Rittle-Johnson and Alibali's (1999) study of mathematical equivalence, children evaluated a set of six different procedures on a three-point scale both before and after a conceptual lesson, or without instruction in a control group. As seen in Figure 3, children in the conceptual-instruction group altered their evaluations in appropriate ways, decreasing their evaluations of incorrect procedures ($M = -0.32$) and increasing their evaluations of correct procedures ($M = +0.20$). In contrast, children in the control group did not change their evaluations of incorrect procedures ($M = +0.01$) and actually decreased their evaluations of correct procedures ($M = -0.20$). These data suggest that gains

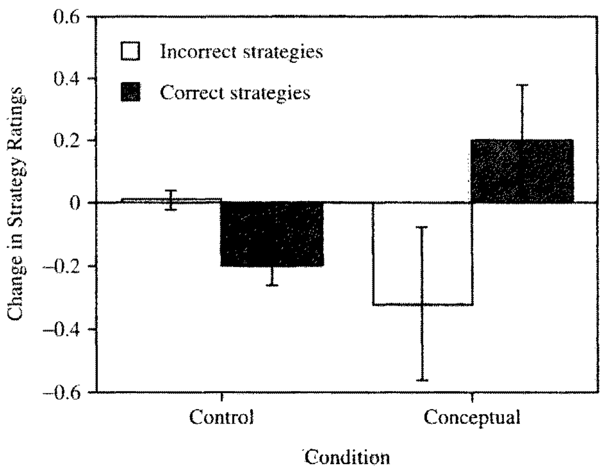

Fig. 3. Mean pretest to posttest change in children's evaluations of correct procedures (N = 3) and incorrect procedures (N = 3) for children in the conceptual-instruction group and the control group. Data drawn from Rittle-Johnson and Alibali (1999).

in conceptual knowledge due to the lesson helped children in the conceptual-instruction group to evaluate potential alternative procedures in sensible ways.

Changes in children's evaluations of alternative procedures presumably inform children's procedure choices. Contemporary models of procedure choice in problem solving, such as that proposed by Shrager and Siegler (1998) (see also Siegler & Shipley, 1995), hold that learners store information about the effectiveness and efficiency of various potential procedures and use this information to guide their procedure choices. Within such a framework, improvements in children's evaluations of alternative procedures, such as increases in evaluations of correct procedures and decreases in evaluations of incorrect procedures, are likely to lead to better choices among competing procedures.

4. *Summary*

Existing empirical data strongly support the possibility that gains in conceptual knowledge lead to changes in problem representation, which in turn enable the generation of new procedures. Less empirical data support the possibilities that (1) gains in conceptual knowledge lead children to realize that existing procedures are incorrect, and this impels them to generate new procedures, and (2) gains in conceptual knowledge guide children's evaluation of alternative procedures, both self-generated ones and ones learned from outside sources. However, the available data suggest that these are also viable pathways to account for the impact of gains in conceptual knowledge on procedural knowledge.

B. PROCEDURES PROVIDE THE BASIS FOR INFERRING CONCEPTS

Mounting evidence indicates that gains in procedural knowledge can influence conceptual knowledge. Several sources of evidence converge on this conclusion. First, several studies have shown that children demonstrate gains in conceptual knowledge after a procedural lesson. For example, Rittle-Johnson and Alibali (1999) provided third- and fourth-grade students with instruction about a correct procedure for solving equivalence problems (cancel like addends and group the remaining addends). Children's conceptual understanding of the equal sign symbol was assessed both before and after the lesson, using a battery of tasks designed to assess understanding both explicitly (e.g., tell what the equal sign means) and implicitly (e.g., rate other children's definitions of the equal sign, evaluate non-standard equations as making sense or not). Children who received the procedural lesson made greater gains in conceptual knowledge than did children in a control group who did not receive any lesson. At the individual level, 53% of children in the procedural instruction group improved their conceptual understanding from pretest to posttest, whereas only 38% of children in the control group did so.

Second, some evidence indicates that improvements in procedural knowledge are associated with gains in conceptual knowledge. As described previously, Rittle-Johnson, Siegler, and Alibali (2001) assessed children's procedural skill at placing decimal fractions on a number line on a pretest, during an instructional intervention, and on a posttest. They also assessed children's conceptual understanding of decimal fractions at pretest and posttest. In a regression analysis, they controlled for scores on the procedural knowledge pretest, and determined that procedural knowledge scores during the intervention and at posttest predicted improvements from pretest to posttest in children's conceptual knowledge. Procedural knowledge scores at both time points were positively associated with size of conceptual knowledge gains. Thus, learning to correctly place fractions on the number line was linked with improvements in children' conceptual knowledge of decimal fractions.

However, despite the evidence that gains in procedural knowledge *can* lead to improvements in conceptual knowledge, some evidence suggests that this causal pathway is less strong or less consistent than the reverse pathway (that from gains in conceptual knowledge to improvements in procedural knowledge). For example, Rittle-Johnson and Alibali (1999) found that more than 80% of children who received conceptual instruction generated correct procedures for solving problems at posttest, whereas only 53% of children who received procedural instruction displayed pretest to posttest gains in conceptual knowledge. Of course, one cannot be certain that the conceptual and procedural knowledge assessments were equally sensitive. However, the pattern of results suggests that the strength of influence of each type of knowledge on the other may be asymmetrical. In addition, many children in the procedural-instruction group were unable to adapt their newly learned procedures to solve transfer problems, suggesting that they had not acquired sufficient conceptual knowledge to guide their adaptation of the instructed procedure to novel contexts. Thus, although children did acquire conceptual knowledge from a procedural lesson, gains were modest and did not hold for all children.

Other evidence also suggests that the influence of gains in procedural knowledge on conceptual knowledge may be limited. In some domains (e.g., multi-digit subtraction, fraction multiplication, fraction division), people learn correct procedures but never fully understand the conceptual underpinnings of those procedures (Fuson, 1990; Ma, 1999). Furthermore, Byrnes and Wasik (1991) provided children a lesson on the least common denominator procedure for fraction addition, but did not observe gains in conceptual knowledge following the lesson. There are several potential explanations for this null result; however, one possibility is that gains in procedural knowledge do not promote gains in conceptual knowledge in all cases or for all children.

This work suggests that any purported mechanisms by which procedural knowledge may lead to gains in conceptual knowledge must be able to account for variability across children in whether or not the gains occur. In the following sections, I consider two potential mechanisms, and review existing empirical evidence for each: (1) learning new procedures may cause children to change their problem representations, and improved problem representation may in turn provide a basis for inferring concepts; and (2) learning new procedures may lead children to reflect upon why those procedures "work" or are effective, and this may provide a basis for inferring concepts.

1. Changes in Problem Representation

One possible mechanism by which gains in procedural knowledge may promote gains in conceptual knowledge is via changes in problem representation. Learning new procedures may cause children to improve their problem representations, and improved problem representation may in turn provide a basis for inferring concepts.

A hypothetical example may serve to clarify the processes involved in this pathway. Consider a fourth-grade boy who holds an operational concept of the equal sign, namely that the equal sign means "put the answer." Suppose that this boy learns that one correct way to solve an equivalence problem such as $3 + 9 + 5 = 3 + \_$ is to cancel the addend that appears on both sides of the equal sign (in this case, the 3), and add the remaining numbers (in this case, the 9 and 5). After learning this procedure, the boy may be more likely to notice the position of the equal sign in other equations that he encounters, because it is essential to attend to the equal sign in order to determine what the "sides" of the problem are. Thus, learning this new procedure may lead him to represent equivalence problems more accurately than he did before. In particular, he may more accurately represent the position of the equal sign.

If the boy begins to accurately represent the position of the equal sign in equivalence problems and other equations, he is likely to note that the equal sign does not always occur at the end of the problem. However, this fact is inconsistent with his conception of the equal sign as meaning "put the answer." Recognizing this inconsistency may lead him to consider other possible interpretations of the equal sign that are consistent with his newly learned procedure. In so doing, he may infer that the equal sign indicates that two quantities are the same—a conception that is compatible with his newly learned procedure. Thus, in this instance, the boy learned a new problem-solving procedure and consequently improved his representation of equations, and his improved representation provided the basis for inferring an improved concept of the equal sign.

No evidence to date addresses this pathway in its entirety. However, there is some evidence for the first link in the pathway, namely, the link from gains in procedural knowledge to improvements in problem representation. Alibali, Ockuly, and Fischer (2005) investigated whether learning new, correct procedures via instruction led to improvements in children's representations of equivalence problems. Third- and fourth-grade students completed reconstruction and recognition tasks to assess problem representation both before and after an intervention in which some children learned a correct procedure for solving equivalence problems. The intervention focused on two different correct procedures, *equalize* (make both sides equal) and *add–subtract* (add the left side and subtract the number on the right), which were presented in a 2×2 design, yielding four groups: *equalize* only, *add–subtract* only, both procedures, or neither procedure. As seen in Figure 4, children who learned the *equalize* procedure improved their problem representations more than children who did not learn the *equalize* procedure. Learning this procedure led children to represent the right side of the problems more accurately (i.e., at posttest, fewer children made errors such as reconstructing the target problem $3 + 9 + 5 = 3 + \_$ as $3 + 9 + 5 = 3$ or $3 + 9 + 5 = +3$).

Surprisingly, learning the *add–subtract* procedure did not lead to improvements in problem representation. This finding suggests a possible reason why not all procedural lessons lead to conceptual knowledge gains: namely, some procedures may be better at promoting improved problem representation than others. Indeed, the *equalize* procedure requires children to represent the *sides* of the equation, whereas the *add–subtract* procedure requires only that children represent the numbers on the left and the number on the right. The fact that the *equalize* lesson led to improvements in children's representations of the right side

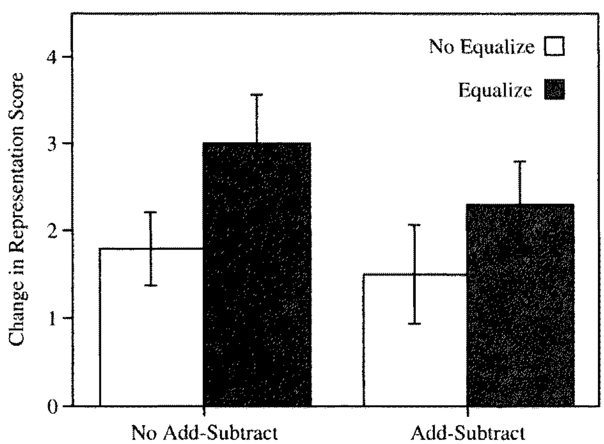

Fig. 4. Mean pretest to posttest improvements in representation scores for children who were taught the equalize *procedure, the* add–subtract *procedure, both procedures, or neither procedure (from Alibali, Ockuly, & Fischer, 2005). Children who learned the* equalize *procedure improved their problem representations more than children who did not learn the* equalize *procedure, $F(1, 45) = 4.2$, $p < 0.05$.*

of the equation is consistent with this view. Thus, some procedural lessons may be more likely to engage this pathway than others.

Unfortunately, there is not direct evidence yet for the second link in the pathway, namely, the link from improvements in problem representation to inferring correct concepts. To address this issue, studies are needed that involve manipulations to improve problem representations and measures of conceptual knowledge both before and after those manipulations. Because the process of inferring new concepts is likely to be gradual, it would seem wise to use sensitive measures of conceptual knowledge, such as rating or recognition measures (e.g., rating correct and incorrect definitions of the equal sign provided by other children) so that subtle changes in concepts can be detected. Such studies might also include probes about concepts in various contexts (e.g., asking about the meaning of the equal sign symbol in various contexts, such as an addition problem and an equivalence problem), because past research has shown that emerging conceptual knowledge may be displayed in some contexts but not others (McNeil & Alibali, in press-a).

2. *Reflection about Why Procedures Work*

Another possible mechanism by which learning new procedures may lead to gains in conceptual knowledge is by fostering reflection about why those procedures work. When children learn new procedures, they may choose to reflect about the rationales underlying those procedures, even if those rationales are not provided in instruction. What might compel such reflection? One possibility is that newly learned, correct procedures yield different solutions than previously used, incorrect procedures. Noting the differences in the solutions yielded by the old and the new procedures may lead children to reflect about the basis of the new procedures. Another possibility is that newly learned procedures may be more efficient than previously used procedures. If this is the case, then children may be able to implement the new procedures without using all of their available processing resources, and they may have resources available which they may allocate to consider the conceptual basis of the new procedures (*cf.* Shrager & Siegler, 1998). Reflecting about newly learned procedures may lead children to articulate principles or to form links between previously isolated pieces of knowledge, both forms of conceptual knowledge. Thus, learning new procedures may provoke reflection about why the procedures work, and this reflection may promote gains in conceptual knowledge.

Consider a girl who initially uses the *add-all* procedure to solve equivalence problems such as $3 + 9 + 5 = 3 + \_$. If the girl then learns the *cancel-and-group* procedure by observing a friend solving the same problem, the girl will note that the new procedure yields a different solution. She may also find that the *cancel-and-group* procedure is easier and less resource-intensive to implement, because it requires adding only two of the addends in the problem, rather than all four. Thus, the girl may have sufficient processing resources available to reflect on the basis of the new procedure, as well as reason to consider why her friend's procedure yields a different solution than her own prior procedure. She may reflect about the implications of canceling like addends, and realize that the new procedure serves to make both sides of the problem equal. This insight may then form the basis for inferring or articulating the principle that the two sides of an equation represent the same quantity.

Some evidence in support of this pathway can be gleaned from existing studies, most notably from McNeil and Alibali's (2000) investigation of the effects of externally imposed goals on children's learning from a procedural lesson. Past research (e.g., Dweck & Elliot, 1983; Licht & Dweck, 1984; Dweck & Leggett, 1988) had suggested that children's achievement goals influence their behavior in learning contexts. With this in mind, McNeil and Alibali (2000) compared the likelihood of making conceptual gains from a procedural lesson for three groups of children: (1) children who were provided a *learning* goal at the outset of the lesson, (2) children who were provided a *performance* goal at the outset of the lesson, and (3) children who were not provided any explicit goal at the outset of the lesson. The goal manipulations were adapted from Elliot and Dweck (1988). The performance goal manipulation focused on solving the problems correctly, and included the phrases, "the most important thing will be for you to try to solve the problems correctly," and "you will be tested on more problems so that I can see how well you can do." In contrast, the learning goal manipulation focused on understanding of the domain, and included the phrases, "the most important thing will be for you to think about the problems and understand them," and "you will do some more problems so you can see how much you've learned." Following the manipulation, all children received a lesson focusing on the cancel-and-group procedure.

Figure 5 presents the proportion of children in each of the three conditions whose scores on the conceptual assessments increased, stayed the same, or decreased from pretest to posttest. As seen in the figure, children in both goal conditions were more likely to increase their scores than children in the no goal (control) condition. However, the proportion of children who increased their scores was greatest in the group of children who received learning goals, which focused on understanding of the domain. The learning goal manipulation likely promoted conceptual gain because it encouraged children to try to understand—and therefore to reflect about—the taught procedure.

Individual children probably differ in their tendency to reflect upon the procedures they use—so this mechanism of change also provides a ready explanation for why some children acquire conceptual knowledge after a procedural lesson, and others do not. The goal manipulations in McNeil and Alibali (2000), and in particular, the learning goal manipulation, may have

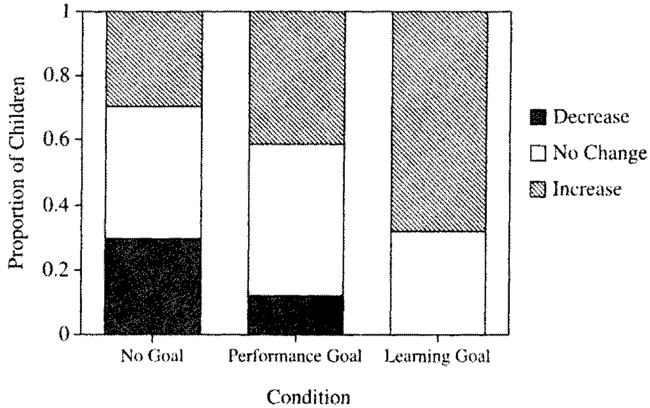

Fig. 5. *Proportion of children in each of three goal conditions whose scores on the conceptual assessments increased, stayed the same, and decreased from pretest to posttest. From McNeil and Alibali (2000). Copyright © 2000 by the American Psychological Association. Reprinted with permission.*

encouraged reflection among children who might not have reflected if left to their own devices.

3. *Summary*

Gains in procedural knowledge clearly *can* lead to improvements in conceptual knowledge; however, such improvements are not observed in all studies or among all children, so the mechanisms underlying this pathway must be able to explain such variability. Empirical evidence indicates that gains in procedural knowledge lead to improvements in problem representation, and it is plausible, though as yet undocumented, that improvements in problem representation may provide learners with a basis for inferring domain concepts. Also, some evidence, albeit limited, suggests that reflection about why procedures work can promote gains in conceptual knowledge. Individual learners may differ in their tendencies to reflect upon or to draw inferences based on new knowledge, so both of these purported mechanisms could account for individual differences in the likelihood of conceptual gains after learning new procedures.

These hypothesized individual differences are reminiscent of reported individual differences in students' tendencies to explain material to themselves as they study—termed "self-explanations" (Chi *et al.*, 1989). A large body of literature suggests that explanations—both when produced for the self and when produced for others—may instigate knowledge change. In Section III, I consider potential mechanisms of knowledge change that involve explanation, both in speech and in gesture.

III. Mechanisms that Involve Expressing Knowledge in Speech and Gesture

Several lines of research have suggested that expressing or explaining knowledge may lead to knowledge change. In the following subsections, I review research on two types of activities that have been linked to knowledge change in past research: (1) producing explanations, for example, when solving problems or studying worked problem examples, and (2) producing spontaneous gestures during explanations. In each subsection, I first review evidence linking the activity to knowledge change, and then consider potential mechanisms that may be responsible for the observed effects.

A. EXPLANATION AND KNOWLEDGE CHANGE

Explanation can be defined as "a family of activities aimed at making something more understandable" (Neuman, Leibowitz, & Schwarz, 2000). Explanations involve activities such as generating inferences, filling in details, articulating underlying principles, generating justifications for problem-solving steps, and so forth. Explanations typically go beyond what was directly stated in the material being explained.

A large body of literature has investigated the role of explanation in learning. The bulk of this literature has focused on two types of explanations: (1) instructional explanations, which are explanations provided by teachers, tutors, or other agents that provide instruction (such as computer-based learning environments), and (2) self-explanations, which are explanations generated by learners themselves. Because my focus in this chapter is on mechanisms that involve expression of knowledge, in this section I consider self-explanations and their role in cognitive change.

The study of self-explanations began with Chi and colleagues' seminal research on college students learning about Newtonian particle dynamics (Chi *et al.*, 1989). This research investigated how students studied a chapter from a physics textbook, and in particular, how students studied the worked example problems provided in the text. Chi *et al.* analyzed the self-explanations that students generated when studying the worked examples and investigated how students' self-explanations related to their learning outcomes. The main finding was that good learners produced more self-explanations of the physics content than did poor learners. Specifically, good learners generated many explanations that refined or expanded upon the conditions for particular actions taken in the examples, extrapolated the consequences of actions beyond those stated in the examples, imposed goals for particular actions or sets of actions, or explained the meaning of quantitative expressions. Thus, the self-explanations produced by good learners contained information about the conditions, consequences, goals, and meanings of the steps taken in the examples.

The fact that good learners produced more self-explanations than poor learners suggested to Chi *et al.* that self-explanations might actually foster understanding of domain principles. To evaluate this possibility, Chi *et al.* assessed the number of components of Newton's laws that students possessed before studying the examples (on a pretest), and the number of components that they mentioned in the explanations they produced while studying the examples. The good learners stated components of the laws during their explanations that they had not mentioned at all on the pretest. Thus, self-explanations served to "bootstrap" students' understanding of domain principles.

Renkl (1997) also highlighted the role of self-explanation in understanding of principles, in a study of university freshmen studying worked examples of probability problems. Renkl coded the content of participants' self-explanations of worked examples along several dimensions, and then used cluster analysis to identify "styles" of self-explanation. Students' learning outcomes varied as a function of self-explanation style, with two style groups showing positive learning outcomes: (1) individuals who used anticipative reasoning, defined as working part of the problem before viewing the worked example, and (2) individuals whose self-explanations focused on principles. Renkl's findings about the beneficial effects of a principle-based self-explanation style are compatible with Chi *et al.*'s claim that self-explanations foster understanding of domain principles.

Renkl's study also replicated Chi *et al.*'s (1989) finding that better learners produced more self-explanations than less successful learners. This finding has also been replicated by other investigators in other participant populations and content domains, including adults learning about computer programming (Pirolli & Recker, 1994) and high school students learning to solve algebra word problems (Neuman *et al.*, 2000).

Early studies of the self-explanation effect were correlational—they linked pre-existing individual differences in explanation patterns to differences in learning outcomes. However, because of the obvious educational implications of self-explanations, intervention studies soon followed. In the first controlled study of whether eliciting self-explanations fosters learning, Chi *et al.* (1994) randomly assigned eighth-grade students studying a text about the circulatory system to self-explain during the study period or to simply read the text twice. Students in the self-explanation condition learned more than students in the control condition. Furthermore, students who generated many self-explanations were more likely to achieve the correct mental model of the circulatory system over the course of the experiment than were students who generated few self-explanations.

A large number of studies in a wide variety of participant populations and content domains have shown that elicited or prompted self-explanations are beneficial for learning. These studies have spanned a number of different types of material to be explained, including text (e.g., Chi *et al.*, 1994), worked examples (e.g., Didierjean & Cauzinille-Marmeche, 1997; Atkinson, Renkl, & Merrill, 2003), problem solutions (e.g., Pine & Messer, 2000), and problem-solving activities (e.g., Neuman & Schwarz, 1998; Aleven & Koedinger, 2002). These studies have also addressed many different content areas, including computer programming (Bielaczyc, Pirolli, & Brown, 1995), analogical problems (Neuman & Schwarz, 1998), block balancing problems (Pine & Messer, 2000), problems about compound and real interest (Renkl *et al.*, 1998), and several mathematical domains, including factoring (Didierjean & Cauzinille-Marmeche, 1997), geometry (Aleven & Koedinger, 2002), mathematical equivalence (Rittle-Johnson, 2004) and probability (Wong, Lawson, & Keeves, 2002; Atkinson, Renkl, & Merrill, 2003). A few studies have yielded null effects (e.g., Mwangi & Sweller's, 1998, study of third-grade students learning to solve "compare" word problems); however, on the whole, this body of literature suggests that eliciting self-explanations is a robust and effective technique for promoting learning and cognitive change.

For elicited self-explanations, the nature of the information explained also influences what and how much explainers learn. Siegler (1995) elicited explanations among 5-year-old children solving Piagetian quantity conservation tasks. Children in one group explained their own, usually incorrect, judgments about the equivalence of the quantities in the tasks, and they were given feedback about the correctness of their judgments. Children in another group were told the correct judgment (e.g., the two quantities were the same) by the experimenter and then explained that judgment. Children who explained the experimenter's reasoning performed better than children who explained their own reasoning.

In a related study, Siegler (2002) elicited explanations among third- and fourth-grade students solving mathematical equivalence problems. Some children were told that a child at another school had provided the correct answer, and they explained why they thought that answer was correct. Other children were told about two children at another school, one who had provided the correct answer, and one who had provided an incorrect answer. The children then explained both why the correct answer was correct and why the incorrect answer was incorrect. Compared to children who explained their own reasoning, children in both groups benefited from explaining other children's answers, but those who explained both the correct and the incorrect answers benefited more than did those who explained the correct answer only. Similar findings were reported by Curry (2004) in a parallel study of college students setting up equations for algebra word problems. Thus, the effects of self-explanation also depend on the information being explained.

Many investigators who have studied self-explanation have theorized about or investigated the mechanisms that underlie the self-explanation effect. In the following sections, I discuss three potential mechanisms that may be responsible for the beneficial effects of self-explanation on learning. Because my focus in this chapter is on changes in mathematical thinking, I emphasize mechanisms that have been studied in the domain of mathematics, although I also consider some studies from other domains. My intent is not to provide an exhaustive review of mechanistic accounts of self-explanation effects, but rather to illustrate some mechanisms that have relevance for mathematics learning. I focus here on three potential mechanisms: (1) self-explaining encourages inference generation, which creates new chunks of knowledge, (2) self-explaining promotes the integration of concepts and procedures, and (3) self-explaining leads to improved problem representation.

1. Generating Inferences

Chi and colleagues' initial claim about the mechanism underlying the self-explanation effect was that self-explanations promote inference generation. Inferences go beyond what was stated in the source material, sometimes drawing on commonsense world knowledge or domain knowledge, sometimes integrating across components of the source text, and sometimes being purely deductive inferences based on statements in the source text (Chi & VanLehn, 1991; Chi, 2000). Inferences may make tacit knowledge (learners' own prior tacit knowledge, or information implicit in the text) more explicit and available for use (Chi & VanLehn, 1991), or they may help solvers to identify and repair gaps in their knowledge of the domain (VanLehn, Jones, & Chi, 1992). Thus, generating inferences is a mechanism by which new knowledge chunks can be created.

Chi and colleagues' view was based on studies of self-explanations produced in the domains of physics and biology. Might a similar process of inference generation also apply in mathematical domains? It seems likely. A hypothetical example can illustrate how inference might lead to new knowledge in a mathematical domain. Imagine a seventh-grade girl studying a worked example of a solved equation, such as the one shown in Figure 6, and providing self-explanations. At Step 3 of the worked example, the girl might notice that 3 has been subtracted from the quantity $3 + x$, and she might notice that $3 + x - 3$ yields x. Based on these observations, she might draw the inference that the purpose of the step is to isolate the variable: "OK, so now they're subtracting 3 from $3 + x$. So I guess the idea is to get the x by itself." In this self-explanation, the girl inferred the goal for the action taken in the worked example. This inference may be a new knowledge chunk for the girl, and it may fill a gap in her understanding of algebraic manipulation.

Is there empirical evidence from mathematical domains to support the view that self-explanations involve generating inferences, which leads to the formation

$$
\begin{aligned}
&1. \quad 3 + 9 + 5 = 3 + x \\
&2. \quad 17 = 3 + x \\
&3. \quad 17 - 3 = 3 + x - 3 \\
&4. \quad 14 = x
\end{aligned}
$$

Fig. 6. Sample worked example of an algebraic equation.

of new knowledge chunks? At least one study of self-explanation in a mathematical domain has yielded such evidence—Wong, Lawson and Keeves' (2002) study of students learning a new theorem in geometry. These investigators provided a group of ninth-grade students with training in self-explanation. These students were then asked to study a textbook section that presented the theorem that an angle inscribed in a semi-circle is a right angle. Students in a control group studied the same textbook section using their usual study techniques. Students in both groups provided think-aloud protocols while studying the text. Finally, all students completed a posttest that included problems that required use of the target theorem for solution.

Students' think-aloud protocols were coded for knowledge-generation activities, defined as statements or actions that involved using given information or prior knowledge to create new connections or relations. Examples include reasoning or hypothesizing that involved the target theorem and relating the target theorem to other theorems. Thus, most knowledge-generation activities involved forming inferences based on given information or prior knowledge. Students in the self-explanation group produced significantly more of these knowledge-generation statements than controls.

The beneficial effects of self-explanation were clear on the posttest, where students in the self-explanation group outperformed students in the control group by a substantial margin, particularly on the more difficult items. Moreover, a path analysis indicated that the effects of self-explanation on posttest performance involved a pathway that included knowledge-generation activities. In fact, knowledge-generation activities were a stronger predictor of posttest performance than any of the other candidate predictors tested, including prior knowledge and beliefs about mathematics. Thus, training in self-explanation fostered students' use of knowledge-generation activities, and these activities promoted success in problem solving.

This study indicates that self-explanations can be used to generate knowledge via inference in at least one mathematical domain. However, the nature of the specific connections formed and how this new knowledge came into play in problem solving were not addressed in Wong *et al.*'s study. Detailed models, such as that developed by VanLehn, Jones, and Chi (1992) for the domain of physics, will be needed to specify the processes involved with more precision.

Future studies should attempt to specify the nature of the new knowledge chunks that are generated via inference. In addition, studies that focus on mathematical domains other then geometry are needed.

2. *Promoting Integration of Concepts and Procedures*

One type of inference that is often evident in self-explanations of worked examples is justifications for particular steps in the solution process (Chi *et al.*, 1989; Chi & VanLehn, 1991). It seems reasonable to assume that self-explanations that focus on such justifications may foster learning by promoting integration of concepts and procedures. Other types of self-explanations may contribute to this integration, as well. For example, explanations that invoke domain concepts may strengthen links between those concepts and procedures that are based on those concepts, even when those explanations do not focus on justifying steps in the procedures.

When concepts and procedures are well-integrated, knowledge is flexible and can be generalized to new tasks and problems (Baroody, 2003). Thus, self-explanations that promote integration of concepts and procedures seem likely to foster learners' acquisition of generalizable, transferable knowledge, and consequently, better problem-solving performance.

Some suggestive evidence from studies of self-explanation in mathematical domains supports this potential mechanism. Rittle-Johnson (2004) examined the effects of prompted self-explanations on third- through fifth-grade children learning to solve mathematical equivalence problems. Children solved equivalence problems and were given feedback on their solutions. Some children were then prompted to explain both an incorrect and a correct solution that had been obtained by "children from another school" (as in Siegler's 2002 study). Children also received either instruction about a correct procedure or encouragement to invent a new procedure.

Children who self-explained performed better than children who did not self-explain on the posttest and on a transfer test that included novel problems (e.g., problems that did not have a repeated addend), regardless of whether they had received direct instruction or encouragement to invent a procedure. To succeed on the transfer test, children needed to adapt their newly learned or invented procedures for the novel problems, and conceptual knowledge is presumably necessary to guide such adaptation. Although children rarely mentioned conceptual information explicitly in their self-explanations, the fact that children who self-explained performed better on the transfer test suggests that these children had constructed connections between their new procedures and their conceptual knowledge of the domain.

Aleven and Koedinger (2002) also provided evidence for the integration of concepts and procedures in their research on self-explanations in geometry learning. Their study utilized a high school geometry curriculum, Cognitive

Tutor Geometry, that involves a computer-based instructional environment. The study focused on a unit about angles in which students learned to solve problems that relied on various theorems (e.g., the alternate interior angles theorem) for solution. Two versions of the Cognitive Tutor were compared: (1) a self-explanation version, which prompted students to self-explain as they worked problems, and (2) the standard, problem-solving version, which did not. In the self-explanation tutor, students provided justifications for each of their problem-solving steps, either by typing in the name of the theorem or principle that justified the step or by selecting the theorem or principle from a glossary. The tutor also provided feedback about whether students' explanations were correct.

Students who used the self-explanation tutor performed better on a problem-solving posttest than did students who used the standard version of the tutor. Furthermore, the advantage of self-explanation was especially pronounced on items that required deeper understanding for success (e.g., items that required students to determine that there was not sufficient information for solution).

To gain a better understanding of students' patterns of success across items on the posttest, Aleven and Koedinger (2002) fit a mathematical model to their data. The model was based on the assumption that performance on the posttest items was due to a mixture of three types of knowledge: (1) shallow procedural knowledge, which included both incorrect procedures and guessing heuristics (e.g., "angles that look the same, are the same"), (2) correct procedural knowledge, defined as knowledge of correct procedures, and (3) conceptual knowledge, defined as well-integrated verbal and visual knowledge about the theorems used in solving the problems. These three types of knowledge were hypothesized to relate to performance on different types of posttest items in different ways. For example, correct performance on easy-to-guess posttest problems could be based on any of the three types of knowledge, whereas correct performance on some of the more difficult items required conceptual knowledge. Aleven and Koedinger formulated equations that captured the hypothesized relations between the three types of knowledge and performance on the different types of posttest items. They then fit these equations to the data, generating estimates of the strength of each of the three knowledge types for students in each of the experimental conditions.

The modeling results suggested that students in the standard condition acquired primarily correct procedural knowledge, whereas students in the self-explanation condition acquired both conceptual knowledge and correct procedural knowledge. These findings are compatible with the idea that self-explanation fostered students' understanding of the conceptual basis of their problem-solving procedures.

Although Aleven and Koedinger (2002) and Rittle-Johnson (2004) have provided suggestive evidence that self-explanation promotes integration of concepts and procedures, neither study has presented direct evidence for this mechanism. Future studies should include assessment items that tap integration of concepts and procedures more directly.

3. Improvements in Problem Representation

Another possible mechanism by which self-explanation may promote learning is via improvements in problem representation. In her research on self-explanations in the domain of biology, Chi (2000) discussed this basic idea in terms of *revising mental models*. To evaluate this potential mechanism, Chi diagnosed eighth-grade students' mental models of the circulatory system before they read and self-explained a text about it. With knowledge of students' initial mental models, it was possible to identify information in the to-be-explained text that would conflict with each student's model. In an intensive microgenetic analysis of one girl's self-explanations, Chi (2000) showed that at the point when the girl detected the conflict between the information in the text and her own mental model, she attempted to resolve the conflict, "resulting in [a] long and tortuous explanation," (p. 214) which appeared to help her repair her initial mental model. The repair process involved adding features that were not present in her original model and integrating links in the model. Thus, self-explaining helped the student to construct a better representation of the circulatory system.

Research on self-explanations of analytical reasoning problems also suggests that self-explanation promotes accurate problem representation. Neuman and Schwarz (1998) found that university students who were prompted to self-explain a set of analytical reasoning problems were more successful at solving a target problem than were students who simply provided think-aloud protocols. A content analysis of the protocols suggested that good solvers (most of whom were in the self-explain condition) focused more on the "deep structure" of the target problem, whereas poor solvers (most of whom were in the control condition) focused more on the problem's surface structure. Neuman and Schwarz interpreted these data as suggesting that self-explaining supports solvers in representing the problem's deep structure. However, it should be noted that their study did not include measures of problem representation independent of problem solution.

At least one study of self-explanation in a mathematical domain has also provided suggestive evidence that self-explanations can lead to improved problem representation. Didierjean and Cauzinille-Marmeche (1997) examined self-explanations produced by ninth-grade students studying worked examples of algebra problems that involved factoring. For participants who had some initial knowledge about factoring, prompts to explain led to better performance on a posttest than either of two control conditions, one in which students simply solved problems and one in which students did nothing between pretest and posttest. Didierjean and Cauzinille-Marmeche distinguished two subgroups of participants in the explanation condition based on their patterns of performance on the posttest, namely, participants who performed well on all of the posttest problems and those who performed well only on problems that were highly similar to the worked examples. For the subgroup that performed well on all of the posttest problems, evidence from participants' verbalizations suggested that they constructed an abstract problem schema that they then applied to the posttest problems.[2] Specifically, students in this group often made remarks that pointed out structural similarities between different problems. Like Neuman and Schwarz (1998), Didierjean and Cauzinille-Marmeche argued that self-explaining led students to represent the problems in terms of an abstract schema that captured the problems' deep structure.

The evidence about improvements in representation from Didierjean and Cauzinille-Marmeche's (1997) research is indirect, because there was no explicit measure of problem representation either before or after the explanation manipulation. To my knowledge, there are no studies in mathematical domains that have directly investigated whether self-explanation promotes changes in representation. However, self-explanations may well foster improvements in students' representations of mathematics problems.

To illustrate, suppose a boy were asked to study the worked example shown in Figure 6, and to provide self-explanations. Suppose further that the boy had used the *add-all* procedure to solve similar equations in a pretest prior to studying the example, and that in a pretest of problem representation, he made several errors typical of children who use the *add-all* procedure, such as reconstructing the given equation $4 + 6 + 7 = 4 + x$ as $4 + 6 + 7 + 4$. When reading step 2 of the worked example, the boy might wonder aloud why the solver in the example did not also add the second 3 in the problem, as he himself would have done when solving this problem. Attempting to explain this solution step may lead him to notice the position of the equal sign in the problem, which he may not have represented correctly before studying the worked example: "Hmm, I wonder why they didn't add the other 3. It has a plus sign after it… Oh! But it has an equal sign in front of it." In this case, being prompted to provide explanations for each solution step impelled the child to improve his representation of the problem by accurately representing the position of the equal sign.

If children improve their problem representations as a result of self-explanations (either prompted or spontaneous), this may lead to other sorts of changes in their knowledge. As discussed in the previous section, improved problem representation may lead to gains in both conceptual and procedural knowledge. With respect to conceptual knowledge, improved problem representation may provide children with a more accurate basis for inferring concepts, as in the example presented previously (p. 93). With respect to procedural knowledge, improved problem representation may foster generation

[2]The other subgroup appeared to focus on the specifics of each worked example. Participants in this subgroup appeared to have solved the posttest problems using case-based reasoning.

of new procedures (e.g., Alibali *et al.*, 1998) or may lead to better choices among existing alternative procedures (Siegler, 2002).

The idea that self-explanation promotes improved problem representation is a plausible one, with suggestive evidence to support it. However, few studies, and none yet in the domain of mathematics, have provided compelling evidence that learners' problem representations do in fact improve as a result of self-explanation. Research on this issue in mathematical domains is needed.

4. *Summary*

Existing data support the possibilities that self-explaining leads to changes in mathematical knowledge because (1) self-explaining encourages explainers to generate new knowledge by making inferences, and (2) self-explaining promotes the integration of concepts and procedures. There is less empirical evidence from mathematical domains to suggest that producing self-explanations leads to improved problem representation, which in turn may lead to gains in both conceptual and procedural knowledge. However, this pathway has received indirect support from studies in other problem-solving domains, so it remains a plausible candidate mechanism worthy of future test in mathematical domains.

B. GESTURE PRODUCTION AND KNOWLEDGE CHANGE

The research on explanation and knowledge change has focused primarily on the *verbal* component of explanations. However, in many cases, people produce gestures as well as speech when they articulate inferences or explain how they solved problems. Might these gestures also play a role in knowledge change? Mounting evidence suggests that they do.

Many previous studies have investigated the gestures speakers produce when explaining their problem solutions, both in mathematical tasks (e.g., Goldin-Meadow, Alibali, & Church, 1993), and in non-mathematical tasks (e.g., Crowder & Newman, 1993). It is now widely accepted that speakers' gestures reveal their understanding of the tasks being explained (e.g., Garber *et al.*,1998; Alibali *et al.*, 1999).

Several studies have shown that gestures index transition periods in development and learning. In particular, discrepancies between gesture and speech in children's task explanations, called "gesture–speech mismatches", are associated with readiness to learn. For example, Church and Goldin-Meadow (1986) found that children who frequently produced gestures that mismatched their speech on a pretest of Piagetian conservation were more likely to learn from a brief lesson about conservation than were children who seldom produced gesture–speech mismatches. Similar findings have been reported in several studies of children learning to solve mathematical equivalence problems (Perry, Church, & Goldin-Meadow, 1988; Goldin-Meadow & Singer, 2002). (For examples of gesture–speech mismatches in children's explanations of mathematical equivalence problems, see Table I.)

However, it is not only mismatching gestures that are associated with knowledge change—any gestures may be important. In a study of children learning the concept of mathematical equivalence, Alibali and Goldin-Meadow (1993) found that children who gestured while explaining problems on a pretest tended to learn more from a brief lesson than children who did not gesture, regardless of whether their gestures matched or mismatched speech. The posttest in this study was a paper-and-pencil test that included both addition problems (e.g., $3 + 9 + 5 = 3 + \_$) and multiplication problems (e.g., $4 \times 2 \times 3 = 4 \times \_$). As seen in Figure 7, non-gesturers tended to succeed only on the addition problems on the posttest, whereas gesturers tended to succeed on both types of problems. Thus, gesture production was associated with deeper learning.

Why is gesture production associated with knowledge change? There are at least two possibilities. First, gesture may be an epiphenomenon. It may reflect speakers' knowledge, but not be directly involved in the process of knowledge change itself. Alternatively, gesture may play a functional role in the process of knowledge change. That is, producing gestures may actually influence the path of learning. In this section, I review evidence that producing gestures plays a functional role in knowledge change.

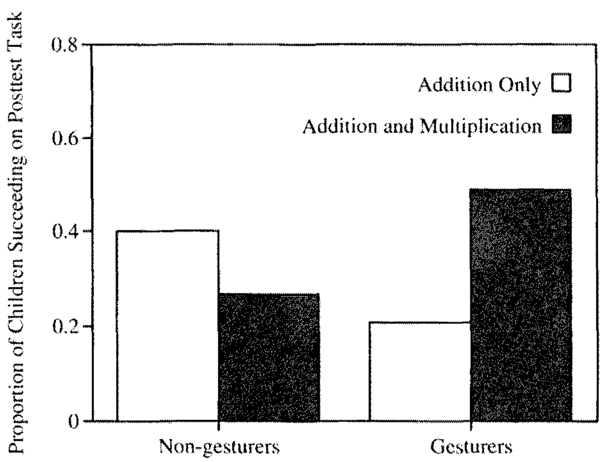

Fig. 7. Proportion of children who produced gestures (N = 43) and who did not produce gestures (N = 15) who succeeded on the paper-and-pencil posttest. Children who produced gestures tended to succeed on both addition and multiplication problems, whereas children who did not produce gestures tended to succeed on addition problems only. Adapted from Alibali, M. W, & Goldin-Meadow, S., Transitions in learning: What the hands reveal about a child's state of mind, Cognitive Psychology, *25, 468–523, copyright © 1993, with permission from Elsevier.*

Research on the function of gesture has focused on several possible mechanisms by which gesture may play a role in knowledge change. Here, I consider two such mechanisms: (1) gesture generates or activates knowledge based in perception and action, making this information available for reasoning and problem solving, and (2) gesture "off-loads" or frees working memory for other purposes, such as procedure generation.

1. Gesture Generates or Activates Knowledge Based in Perception and Action

One pathway by which gesture may play a role in knowledge change is by generating or activating knowledge based in perception and action, making this information available for other cognitive processes. A great deal of evidence suggests that speakers' gestures highlight perceptual and motoric information for *listeners* (e.g., Graham & Argyle, 1975; Kendon, 1994; Valenzeno, Alibali, & Klatzky, 2003). Here, I consider evidence that producing gestures may highlight such information for speakers themselves. If gesture production leads speakers to focus on particular types of information, then speakers who produce gestures should traverse a different path in learning than speakers who do not gesture.

Gestures often express information that is based in perception or motor actions. For example, when a speaker talks about a cup, the accompanying gestures may represent perceptually salient properties of the cup (e.g., size, shape, or height) or information about how the speaker could physically manipulate the cup (e.g., by picking it up, tilting it, or drinking from it). Producing such gestures may activate this perceptual and motor information so that it is readily available for reasoning and problem solving. As a consequence, speakers who produce gestures may tend to focus on perceptual and motor information, rather than other sorts of information, in their problem solutions and problem explanations.

Several lines of research support the claim that gesture promotes a focus on perceptual-motor information. First, gesture appears to help speakers activate images or maintain them in memory. In one study of this issue, de Ruiter (1998) presented speakers with arrays of shapes and lines and asked them to describe them to a listener. For some speakers, the images remained visible during the descriptions, and for other speakers, the images were removed during the description. Speakers produced more gestures when the images were no longer visible, suggesting that gestures helped them to retrieve the images and maintain them in mind. These findings were replicated by Wesp and colleagues, using still-life paintings instead of arrays of shapes (Wesp *et al.*, 2001).

Second, when speakers are prevented from gesturing, they focus less on perceptual or motor information. However, it should be noted that most gesture-prohibition studies have been conducted in non-mathematical domains. For example, Rimè and colleagues examined the content of spontaneous conversations held by speakers who were allowed to gesture and speakers who were prohibited from gesturing (Rimè *et al.*, 1984). The content of the speech was assessed using a "computer program of content analysis conceived to quantify the degree of speech imagery" (p. 317). When gestures were prohibited, speakers received lower imagery scores, supporting the view that gesture promotes a focus on perceptual information. However, this conclusion is necessarily tentative, because Rimè *et al.* provided little information about their content analysis program.

Stronger evidence on this issue was provided by Alibali, Spencer, and Kita (2005), who studied the procedures adults use to solve gear movement prediction problems with gesture allowed and with gesture prohibited. In gear movement prediction problems, participants imagine an array of gears described by the experimenter, and determine how a particular gear would move if another gear were moved in a certain way (e.g., "Imagine five gears are arranged in a circle. If you try to turn the gear on top clockwise, what would the gear just to its left do?"). Past research had shown that solvers frequently used gestures to model the movements of each individual gear as they attempt to solve the problems (Schwartz & Black, 1996).

Alibali *et al.* found that participants who were allowed to gesture tended to use depictive procedures, in which they described the movement of each individual gear, throughout the set of problems. In contrast, participants who were prevented from gesturing often generated rule-based procedures (most often the parity rule, which holds that if there are an odd number of gears, the last gear goes the same direction as the first, and if an even number, the last gear goes in the opposite direction). Gesture appeared to help solvers to mentally simulate the actions of each gear, and therefore promoted use of depictive procedures. Thus, gesture promoted reasoning based on motor actions rather than abstract rules. Speakers solved the problems correctly at comparable rates in both conditions, but gesture influenced their choice of problem-solving procedures.

Only one study to date has examined gesture and perceptual-motor reasoning in a mathematical domain. Alibali, Kita and colleagues (Alibali *et al.*, 2001; Alibali & Kita, 2005) studied the justifications that first- and second-grade students provided for Piagetian quantity conservation tasks when they were allowed to gesture and when they were prohibited from gesturing. All children solved and provided justifications for two sets of conservation tasks. During the first set, all children were allowed to gesture; for the second set, children were randomly assigned to either a gesture-allowed or a gesture-prohibited condition. When gesture was allowed, children focused primarily on perceptual characteristics of the task objects (e.g., the heights of the glasses of water, the lengths of rows of checkers) in their verbal explanations of the tasks. When gesture was prohibited, children were more likely to invoke information that was not perceptually present in the display, such as the initial equivalence of the quantities or the transformation that had been performed.

These findings suggest that, when gesture is allowed, perceptual information is highlighted as children attempt to formulate an explanation for the conservation task. When gesture is prohibited, perceptual information is less salient, so children are more likely to focus on non-perceptual information in formulating their responses. From this study, it is not clear whether prohibiting gestures influenced children's representations of the tasks themselves, or the aspects of those representations that children chose to verbalize. However, in either case, producing gestures appeared to play a role in children's thinking.

But *how* might gesture contribute to thinking? According to Kita (2000), speakers use gestures to explore alternative ways of organizing perceptual and motoric information in planning spoken utterances, in an effort to find a conceptualization that can readily be expressed in speech (see also Hostetter and Alibali, 2004)). I suggest that this exploration may lead to changes in the speaker's mental representation of the problem at hand. In the conservation task, children believe that the objects contain information relevant to the experimenter's question, so when gesture is allowed, children use gesture to explore the physical characteristics of the objects as they attempt to represent the situation and formulate their responses. As a consequence of this exploration, perceptual information should be central in children's representations of the tasks when gesture is allowed. When gesture is prohibited, children cannot use gesture to explore the perceptually present array, so other information about the tasks should be relatively more salient. This is exactly the pattern of data observed by Alibali and Kita in children's verbal explanations.

This view implies that children's problem representations should differ when gesture is allowed vs. when gesture is prohibited. Furthermore, if producing gestures provokes changes in children's problem representations, it may ultimately lead to changes in children's procedural and conceptual knowledge, as described in the first section of this chapter. Future studies that directly test these predictions are needed.

Are there other mathematical domains besides quantity conservation in which gesture might influence reasoning and problem solving? Gestures are especially likely to accompany speech that has spatial or imagistic content (Lavergne & Kimura, 1987; Hadar & Krauss, 1999), so mathematical domains that involve visual imagery, such as geometry, may be particularly likely to elicit gestures. Geometry problems often involve reasoning with images, and using gestures in reasoning about or explaining such problems might promote accurate representation of the problem content.

Likewise, solving story problems often involves forming a mental image of the problem situation, and producing gestures could help speakers to envision such situations more accurately. This possibility gains support from a study of adults solving algebra story problems dealing with discrete and continuous change (Alibali *et al.*, 1999). Participants were asked to read story problems silently, and then explain the gist of each problem to another participant. Speakers produced many gestures when describing the problems. After describing each problem, participants were asked to solve that problem. Speakers who produced gestures that conveyed discrete, incremental change during their problem descriptions tended to use additive procedures to solve the problems, whereas speakers who produced gestures that conveyed smooth, continuous change during their problem descriptions tended to use multiplicative procedures for solving the problems. The gestures speakers produced while describing the problems may have influenced their mental representations of the problems, and consequently, influenced their solution procedures.

In sum, there is growing evidence that gesture promotes a focus on perceptual and motor information. To date, most studies of this function of gesture have focused on non-mathematical tasks, such as the gear movement prediction task. It seems probable that gesture could also influence problem representation in mathematical domains by highlighting perceptual information, particularly for tasks that involve a perceptual component. However, research directly addressing this issue is needed.

2. Gesture Off-Loads Working Memory

Another possible mechanism by which gesture production leads to knowledge change is by off-loading working memory, so that it can be used for other purposes, such as procedure discovery.

A number of researchers have suggested that gestures provide support for working memory. In one study of this issue, Alibali and DiRusso (1999) investigated the role of gesture in preschoolers' counting. Children counted more accurately when they gestured to the counted objects themselves than when they were prohibited from gesturing or when a puppet gestured for them as they produced the number words. Alibali and DiRusso (1999) argued that gestures serve to externalize some of the contents of working memory, so they need not be held internally. They further proposed that keeping track of counted objects and tagging each object with a number word require fewer working memory resources when the objects are marked physically, with gestures, than when the objects are marked visually, by looking at each item (as when gesture was prohibited or when the puppet gestured). When children gestured to each of the counted objects themselves, they counted more successfully.

Other researchers have explored the role of gesture in managing the memory demands of problem explanation. Goldin-Meadow *et al.* (2001) asked two groups of participants to explain mathematical problems while remembering sets of words or letters. Nine- and ten-year-old participants solved and explained mathematical equivalence problems, and adult participants solved and explained factoring problems of the form $x^2 + 5x + 6 = (?)(?)$. Participants in both groups were prohibited from gesturing during some of their problem explanations and free to gesture during other problem explanations. Both child and adult participants remembered more of the words or letters when they gestured than when they did not. Goldin-Meadow *et al.* interpreted these findings as showing that gesture off-loads working memory, freeing up capacity for the secondary task of remembering words or letters. In essence, gesture "lightens the load" of explanation.

A possible alternative account of Goldin-Meadow *et al.*'s findings is that, instead of gesture lightening the load, being prohibited from gesturing may actually *add* to speakers' load. If this were the case, the findings would be due, not to the beneficial effects of gesture, but to the deleterious effects of being still. To address this possibility, Goldin-Meadow *et al.* reanalyzed the data from a subset of participants—those who chose not to gesture on some of the problem explanations for which they were allowed to do so. The effect of not gesturing was the same whether participants refrained from gesturing spontaneously or did so in response to the experimenter's instructions. Both child and adult participants remembered more of the words or letters when they gestured than when they did not gesture, whether by choice or by instruction. Furthermore, these findings could not be attributed to differences in the difficulty of solving the problems. The percentage of problems solved correctly did not differ as a function of whether participants gestured, did not gesture by choice, or did not gesture by instruction during the problem explanations.

These findings were replicated by Wagner, Nusbaum, and Goldin-Meadow (2004) with the factoring task as the primary task, and with two different secondary tasks: a verbal memory task (remembering lists of letters) and a visuo-spatial memory task (remembering the locations of dots on a grid). In both cases, participants performed better on the memory task when they produced gestures during their problem explanations.[3] Furthermore, as in the prior study, not gesturing spontaneously had the same effects as not gesturing in response to instruction—in both cases, participants remembered less when they failed to gesture.

The findings of Goldin-Meadow *et al.* and Wagner *et al.* suggest that gestures off-load working memory, freeing capacity for other tasks. But how might

[3]In light of research showing that gesture promotes a focus on perceptual information (Alibali & Kita, 2005; Alibali, Spencer, & Kita, 2005), it seems surprising that prohibiting gesture had a comparable impact on verbal and visuo-spatial memory. One possible resolution of this puzzle has to do with differences in the types of gestures produced across the studies. The factoring task studied by Wagner *et al.* elicited primarily pointing gestures that referred to the numbers on the blackboard. In contrast, the conservation and gears tasks studied by Alibali and colleagues elicited primarily iconic gestures, which depict information via handshape or motion. It is possible that different types of representations underlie pointing and iconic gestures. Moreover, it seems likely that different types of gestures serve different cognitive functions. However, testing these possibilities remains a task for future research.

off-loading working memory be involved in knowledge change? One possibility is that, if resource demands of a primary task (e.g., explanation) are minimized, resources can be used for activities that lead to knowledge change, such as forming more accurate representations of the problems to be solved, or discovering better, more efficient procedures (e.g., Shrager & Siegler, 1998).

If this is the case, then gesture prohibition should make procedure discovery less likely. However, as described previously, in a non-mathematical task, Alibali, Spencer, and Kita (2005) found just the opposite—when speakers were prohibited from gesturing, they were *more* likely to discover a rule-based procedure for solving gears problems. In this study, speakers who were free to gesture perseverated in their use of a depictive procedure. These findings highlight that many factors contribute to processes of procedure selection and procedure change. Producing gestures may decrease the resource demands of explanation, but the availability of gesture for physical modeling may make participants unlikely to shift away from the depictive procedure.

3. Summary

Empirical evidence provides tentative support for at least two mechanisms by which gesture production is involved in knowledge change. Gesture promotes a focus on perceptual and motor information, and may thereby influence problem representation. There is evidence for this mechanism in the literature, although not primarily in mathematical domains. Alternatively, or in addition, gesture may serve to off-load working memory. There is evidence from multiple mathematical domains that gesture serves this function; however, the role of this function in knowledge change needs to be better specified.

IV. Concluding Remarks

In this chapter, I have considered two broad classes of change mechanisms that have been proposed to apply in the domain of mathematical reasoning: (1) mechanisms that involve reciprocal relations between knowledge of problem-solving procedures and knowledge of concepts, and (2) mechanisms that involve expressing knowledge in speech and gestures. Within each class, several candidate mechanisms were described, the empirical evidence for each was reviewed, and gaps in the research base addressing each mechanism were identified.

In this final section, I address two issues. First, are there any core similarities among the mechanisms discussed in this chapter? I consider two characteristics that are shared by many of the mechanisms: (1) the role of the learner's own activity, and (2) a focus on problem representation. Second, is it possible to predict which mechanisms will apply in particular situations? I discuss the possibilities that some mechanisms are better suited for certain types of content or for certain types of learners. I also consider whether there are developmental differences in which mechanisms apply.

A. SIMILARITIES AMONG MECHANISMS

The present review of change mechanisms provokes an obvious question: Are there any similarities that are shared by many or all of them?

1. Emphasis on Learner's Activity

One notable similarity is that many of the mechanisms discussed herein depend on the learner's own activity. Some of the mechanisms require overt actions on the part of the child, such as producing self-explanations or producing gestures. Many of the mechanisms involve constructive mental processes, such as generating inferences (either based on improved problem representations, or as a result of self-explanation), reflecting about why procedures work, forming links between concepts and procedures, and mentally simulating actions or images with the support of gestures. Knowledge change via these mechanisms is not a passive process, but instead requires active processing on the part of the learner.

However, active processes are ones that learners can choose to engage in or not to engage in. Indeed, even when prompted to engage in active processes, individuals differ in how much they do so. For example, Chi *et al.* (1994) found that, among a group of learners who were prompted to provide self-explanations of a text about the circulatory system, some provided more extensive self-explanations than others, and consequently learned more.

Given the central role of active processing in many mechanisms of knowledge change, one key issue for future theories of knowledge change is explaining why some learners are more likely than others to engage in active processing. Better understanding of this issue should shed light on individual differences in learning outcomes. In particular, variations in learners' own activities may be associated with variations in learning from instruction.

2. Emphasis on Problem Representation

Another notable similarity across many of the mechanisms discussed herein is an emphasis on problem representation. Many of the mechanisms involve changes in representation: gains in conceptual knowledge lead to improved problem representation, which may instigate procedure generation; gains in procedural knowledge lead to improved problem representation, which may provide a basis for inferring concepts; self-explanations may lead to improved representation of the problem domain, which may lead to gains in both conceptual and procedural knowledge; and producing gestures promotes a focus on certain types of information in learners' problem representations.

In addition to its role in mechanisms discussed in this chapter, problem representation may also play a role in other change mechanisms, not discussed herein, that involve problem-solving performance. Problem representation is a key determinant of problem-solving performance (e.g., Siegler, 1976; McNeil & Alibali, 2004), and performance plays a crucial role in altering knowledge in some models of developmental change (e.g., McClelland, 1995; Shrager & Siegler, 1998). According to such models, each time a problem is solved, the results feed back to influence the learner's knowledge about that class of problems. Thus, because problem representation influences how problems are solved, changes in problem representation are a potentially powerful engine of knowledge change.

B. WHICH MECHANISMS APPLY WHEN?

Given that there are many possible mechanisms of knowledge change, is it possible to predict which mechanisms may apply in particular situations or for particular individuals?

1. Differences across Content Domains

It seems probable that certain mechanisms are more likely to be engaged in some content domains than others. Much of mathematical learning involves solving problems, so mechanisms that involve procedures may be particularly relevant in mathematical domains. Some of the mechanisms discussed in this chapter are triggered by learning new procedures (e.g., learning new procedures leads to improvements in representation, and improved representations are a basis for inferring concepts). Other mechanisms involve procedures in other ways (e.g., self-explanation of mathematical content promotes the formation of links between concepts and procedures). More generally, knowledge change in mathematics may be particularly likely to engage mechanisms that involve procedures, either as a starting point for change or as an object of change.

It may also be the case that certain sub-domains within mathematics are particularly well suited for certain mechanisms and not others. For example, geometry is a highly visual domain, so mechanisms that involve gesture production may be especially likely to apply in learning about geometry.

2. Individual Differences

It also seems highly probable that individuals differ in the particular mechanisms that they engage. Unfortunately, little is known about how individual differences in cognitive skills, learning styles, personality traits, and other characteristics influence the mechanisms that individuals use. However, it is easy to generate hypotheses about what mechanisms might be likely or unlikely to apply depending on individual characteristics. It seems likely, for example, that individual differences in verbal skill might influence frequency of self-explanation, which has consequences for several mechanisms of knowledge change. Similarly, it is easy to imagine that individual differences in learners' propensities to produce gestures might influence whether or not learners engage mechanisms that involve gesture production. However, to date, neither of these hypotheses has been tested.

The literature does include some hints about dimensions of individual differences that may be associated with tendencies to apply particular change mechanisms. McNeil and Alibali (2000) addressed one of these dimensions—namely, variations in the goals that children bring to the learning situation. In their study, children who were provided learning goals (e.g., understand the problems) were more likely to display conceptual gains in response to procedural instruction than children who were provided with performance goals (e.g., do well on the problems).[4] These findings suggest that children who enter a learning situation with learning goals (regardless of whether those goals are intrinsic to the child or imposed from the outside) may be particularly likely to reflect about why procedures work. Studies such as this one represent a tentative first step toward addressing how mechanisms of change depend on individual differences. This issue is an important arena for future work.

3. Developmental Differences

It is also worth considering whether there are developmental differences in which mechanisms apply and when. It is possible that different mechanisms are most commonly engaged at different ages. However, little research has addressed this issue directly. No studies to my knowledge have compared whether children of different ages are differentially likely to acquire conceptual knowledge after learning new procedures, or differentially likely to benefit from self-explaining.

Most research on mechanisms that involve reciprocal relations between concepts and procedures has been conducted with elementary school children. In their early school years, children learn many new concepts and procedures, and the domains children are learning about (e.g., mathematical equivalence, decimal fractions) are tractable for study. However, learning of concepts and procedures is not limited to the elementary school years. For example, children learn procedures and concepts related to counting in the preschool years. Research on early counting has also focused on relations between concepts and procedures, specifically on the possibilities that preschoolers infer counting principles from implementing the counting procedure (e.g., Briars & Siegler, 1984), and that preschoolers' knowledge of counting principles informs their implementation

[4]This difference between the learning and performance goal groups was observed at the immediate posttest. Two weeks later, at a follow-up assessment, there were no differences between the groups in the proportion of participants who made conceptual gains.

of the counting procedure (e.g., Gelman & Meck, 1983). It seems likely that the reciprocal relations between concepts and procedures that have been observed in various domains are a consequence of how knowledge is organized, rather than due to some characteristic of child learners *per se*. If this is the case, then the same mechanisms should apply in learners of any age.

Research on mechanisms that involve expressing knowledge in language and gesture has been conducted with both children and adults. The benefits of self-explanation have been documented in elementary school children (e.g., Pine & Messer, 2000; Rittle-Johnson, 2004), high school students (e.g., Aleven & Koedinger, 2002; Wong, Lawson, & Keeves, 2002) and adults (e.g., Chi *et al.*, 1989; Renkl, 1997), and the effects of producing gestures on procedure use have also been documented both in children (Alibali & Kita, 2005) and in adults (Alibali, Spencer, & Kita, 2005). Of course, in order for explanations or gestures to influence learners' knowledge, those learners must be old enough to produce explanations or gestures in the first place. However, once learners are able to do so, it seems likely that the mechanisms should apply regardless of age.

The possibility that certain change mechanisms apply regardless of age has not been directly tested for any of the mechanisms discussed in this chapter. Furthermore, because individuals of different ages are typically learning different things, age and content domain are likely to be confounded, so it would be difficult to conduct a strong test of the invariance of change mechanisms across development. However, despite the challenges inherent in testing such claims, research on this issue would be valuable.

4. Multiple Mechanisms

Finally, it is important to acknowledge that in real-world learning situations, multiple mechanisms of change are certainly in play at any given time. For example, consider a girl who learns a new problem-solving procedure during a lesson in her mathematics class. The girl may begin to represent features of the problem that she had not previously noticed. In addition, learning the new procedure may provoke her to self-explain as she solves practice problems, and her self-explanations may incorporate gestures as well as speech. Thus, she may engage multiple change mechanisms simultaneously, and the mechanisms may feed into and reinforce one another. At the same time, the boy sitting at the desk beside her, who experiences the same procedural lesson, may engage a different set of mechanisms, and he may learn something entirely different from the lesson.

Understanding knowledge change in real-world settings will require an appreciation of the simultaneous and interactive nature of mechanisms of change, as well as the considerable variability that is manifested across content domains and across individuals. Although these issues present substantial challenges, a better understanding of underlying mechanisms also promises substantial payoffs, both for advancing theory and for informing practice.

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